Repulsive Casimir Forces & Topological Insulators

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Abstract

Applying boundary conditions to fluctuations of the electromagnetic field results in boundary-dependent modification of the zero-point energy and hence, also a force on the boundaries. This force is known as the Casimir force and this review describes its origins and multiple mathematical descriptions, which can be harnessed in order to calculate the force between any two bodies of any material with known optical properties. Predicted a long time ago, it is in theory possible to modify the boundary conditions in a manner, which creates a repulsive force, as opposed to the usual attractive force, which is found in the vast majority of setups. Different approaches to achieve a repulsive Casimir force are discussed and compared, including forces due to boundary conditions, due to usage of metamaterials, and most importantly – due to usage of three-dimensional topological insulators. Special attention is paid to the latter, including a summary of the most important breakthroughs related to this new class of materials. It is concluded that even though the realisation of repulsive Casimir forces has been predicted in many systems and can be beneficial for a number of applications, an undisputed piece of experimental evidence is yet to be presented. Finally, it is pointed out that a better modelling of topological insulators and their electric/optical properties could lead to an understanding of the bulk and surface contributions to the force, which is presently missing from the field. It is very likely that a breakthrough of this sort will inspire and kick-off a series of experiments, dedicated to measuring a repulsive Casimir force in topological insulators.

Contents

I. Introduction

Imagine that you are a computer chip manufacturer and you create a device, which includes a silicon cantilever. The cantilever itself acts as a clock generator – a driven oscillator, analogous to a heartbeat of the device, which introduces the basic time step of computation. For a large number of oscillations the device functions perfectly but all of a sudden it breaks irreversibly. What happened? An effect of stiction took place, which effectively glued the silicon cantilever to a nearby component, and the culprits are no other than the random electromagnetic fluctuations, constituting the Casimir force.

This example is one of many in modern day electronics, where downsizing is essential for increase in computational power of commercial devices but it comes with new unique hurdles. For instance, at small sizes and separations the finite atomic size and deviation from the thermodynamic limit become more apparent and chaotic behaviour can dominate. Therefore, it is important that these effects are studied for the successful development of micro/nanoelectro-mechanical systems (MEMS/NEMS) among other applications.

Furthermore, the theoretical and experimental study of Casimir forces is of crucial importance to the fundamental comprehension of Physics. The Casimir force involves and arises from the zero-point energy (ZPE) of free space and its connection to photon-matter interactions. Introducing a pair of conducting slabs reduces the amount of available modes between the slabs because they have to obey additional boundary conditions. Consequently, the zero point energy in the region is reduced and this reduction is slab-separation-dependent but more on that later. The ZPE finds roots in the uncertainty principle and the quantisation of energies in quantum systems. It has been and still is one of the biggest problems in Physics for over one hundred years ever since Planck postulated the theory of quantum mechanics. The Casimir forces offer a unique chance to study this phenomenon directly by subjecting micro-sized objects to these forces under different condition, unlike the Lamb shift, for instance, which is also rooted in the ZPE, but can only be probed indirectly by spectroscopic methods.

This review summarises the most important mathematical models, which are used to study Casimir forces. It is then discussed how these models can create desirable effects in various systems. Particular focus is paid to the relatively new class of materials, known as topological insulators (TIs), whose unique electronic properties can result in effects such as tunable Casimir forces. A review of the relevant theoretical breakthroughs is presented as well as a few suggestions for the future of the field.

II. A Theory of Lifshitz

In 1873 Johannes van der Waals published his doctoral dissertation, in which he gave the mathematical description of what he thought was a more realistic scenario for a gas behaviour[1]. Namely, he acknowledged that individual gas particles take up a certain finite volume and additionally, they would experience attractive forces between themselves. Van der Waals's motivation was adding the ability of gases to aggregate in a phase-transition-like fashion to the already existing ideal gas model. The Dutchman had published his dissertation around the same time when the Classical electrodynamics theory had been finalised by Maxwell, among others. This allowed for Lebedev in his own respective doctoral dissertation to postulate that these inter-particle forces may arise from charge fluctuations and the particles would be analogous to little microscopic antennae[2]. Within forty years, neutral particle interactions had been characterised and shown to obey an energy potential dependence $\propto r^{-6}$, where r stands for particle separation and is assumed large compared to the particle size. Most famously, the London dispersion forces fall in that category[3,4] but also Debye[5] and Keesom[6] forces. One way to explain the inverse power six law is in the paradigm of quantum mechanics, where a dipole-dipole interaction energy can be treated in second-order perturbation theory, effectively squaring the r^{-3}-dependent electric dipole potentials[7,8].

In 1948, another Dutch physicist by the name of Hendrick Casimir used the newly developed Quantum field theory (QFT) to derive a force that appears on two infinite conducting parallel plates as a result of field fluctuations[9]. Later, he and Polder published a generalisation on the Van der Waals forces, which included retardation due to the finite speed of light, and in the limit of which the potential scales as r^{-7}[10]. In the last paper the authors express the possibility for the problem to be approached in a more fundamental fashion, thus, providing a general theory for the interactions due to charge and electromagnetic field fluctuations between any two bodies. However, Casimir and Polder never managed to come up with such a complete and rigorous theory. Soon enough, in 1954 Evgeny Lifshitz published a theory, derived from electrodynamics' stress tensor[11], later reformed with his colleagues in the Quantum field theory formalism (DLP theory)[12], which was widely accepted. The theory finds its basis in fundamental electrodynamics and builds on top of that to give the general forces that arise between any two electrically responsive bodies. Furthermore, it reduces to the Casimir, London, Debye, etc. forces in their respective limits and geometries. The theory was later reformulated by usage of summation over multiple modes of effective oscillators within the materials, where boundary conditions arise from the existence of material interfaces, in a series of papers from 1968 to 1976[13–15]. The newer formulation is considered rigorous by some reviewers[16] and rather heuristic by others[17]. In all approaches, the energy of a system due to charge and electromagnetic field fluctuations is shown to be derivable from and dependent solely on the optical properties of the materials and the geometry of the particular setup.

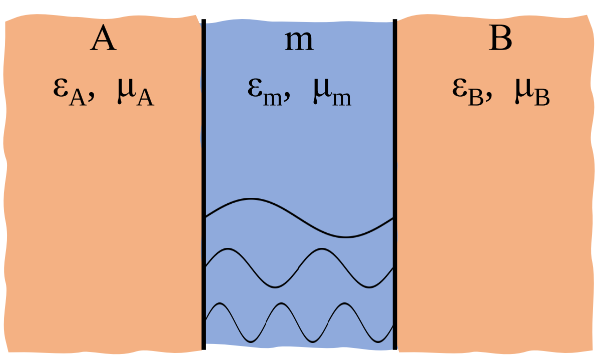


Figure 1: Simplest case for calculation of Van der Waals forces. Two semi-infinite condensed bodies, separated by an arbitrary fluid medium.

The simplest case where two semi-infinite pieces of materials A and B are separated by a distance l and the separation is filled by material m, is depicted in Figure 1. In the heuristic theory, the free energy per unit area it takes to bring the two bodies from infinite separation to the specified separation is expressed as

$$G(l) = \frac{k_B T}{2\pi c^2} \sum_{n=0}^{\infty}{}^{\dagger} \epsilon_m \mu_m \xi_n^2 \int_1^\infty p \times \ln\left((1 - \bar{\Delta}_{Am} \bar{\Delta}_{Bm} e^{-r_n p})(1 - \Delta_{Am} \Delta_{Bm} e^{-r_n p}) \right), \tag{1}$$

with the force per unit area experienced by each condensed body following from $F(l) = \left(\frac{-\partial G(l)}{\partial l}\right)_T$ and being easy to compute in the given logarithmic notation. DLP theory derives the force directly from the electromagnetic stress tensor and sidesteps the expression for energy but the two results are identical. In Equation 1, aside from the regular physical constants, T represents absolute temperature, $r_n = \frac{2l\,\xi_n \sqrt{\epsilon_m \mu_m}}{c}$, $\bar{\Delta}_{ij} = \frac{s_i \epsilon_j - s_j \epsilon_i}{s_i \epsilon_j + s_j \epsilon_i}$ with $s_i = \sqrt{\frac{\epsilon_i \mu_i}{\epsilon_m \mu_m} - 1 + p^2}$, and Δ being the magnetic equivalent of $\bar{\Delta}$ (so that $\epsilon_i \to \mu_i$). For the majority of nonmagnetic materials Equation 1 can be simplified by considering $\mu \approx 1 \forall$ regions but as will be shown later, it is important to consider the magnetic contribution for effects such as a repulsive Casimir force. The integration parameter p is used to simplify the expression. Other than that the integration is over all physically allowed frequencies of an electromagnetic wave, carrying information of the surface mode of one material to that of the other. The dagger on the summation signifies that the first term ($n = 0$) is multiplied by $\frac{1}{2}$[a] and the summation itself is over all $n \in \mathbb{N}$, as defined in the so-called imaginary frequencies $\hbar\xi_n = 2\pi k_B T\, n$. Understanding these imaginary frequencies is the crux of the Lifshitz theory.

In the original DLP theory, they arise naturally as allowed frequency modes of the Fourier series decomposition of a Green's function, corresponding to the free photon, which mediates the electromagnetic forces between the two bodies[12]. Obtaining

[a] Due to the nature of the ground state of electromagnetic fluctuations (harmonic oscillator).

a physically measureable quantity from an interaction amplitude in Quantum field theory always requires integration over all possible 4-momenta. In the DLP paper, it is shown that the aforementioned integration of income/outcome parameters in QFT is equivalent to taking the sum over all imaginary frequencies in DLP theory. In the heuristic theory the imaginary frequencies arise as the allowed electromagnetic modes, which obey the boundary conditions, arising at the material interfaces. With that being said, the dielectric constant and the magnetic permeability in Equation 1 are part of the sum and evaluated at these imaginary frequencies ξ_n. Fortunately, these can be easily related to the imaginary part of the frequency-dependent complex dielectric constant $\epsilon(\omega) = \epsilon'(\omega) + i\epsilon''(\omega)$ by a Kramers-Kronig transform[18]

$$\epsilon(i\xi_n) = 1 + \frac{2}{\pi}\int_o^\infty \frac{\omega\,\epsilon''(\omega)\,d\omega}{\omega^2+\xi_n^2}\,. \tag{2}$$

It becomes clear that the forces, experienced by the two semi-infinite condensed bodies, are then dependent only on the separation and the optical properties of each of the materials. In the heuristic theory it is more intuitive to see that the energy in Equation 1 arises purely between the two interfaces. Hence, adding interfaces will result simply in additional terms in the expression, one per each unique pair of interfaces with variable separation. Different geometries can then be approximated by considering collections of surfaces and taking their limit to smoothen them out. It is quite important to point out that in these theories, it is assumed that the surfaces are flat and continuous – individual atoms are seen only as part of a continuous whole and there is no roughness. This restricts the theory to only be accurate for separations up to $l \sim 5\,\mathrm{nm}$. Later works show both theoretically and experimentally the effect that surface roughness plays on the Casimir and Van der Waals forces at both small and large separations[19–21]. The contribution due to roughness is usually computed with perturbation theory and as can be expected, becomes more significant at smaller separations.

In order to see what the effect of retardation on the system is, it is convenient to make an approximation, in which the magnetic contribution is neglected and $\epsilon_i - \epsilon_j \ll 1\,\forall\, i \neq j,\, \omega$. A Taylor expansion of the expression in Equation 1 around $\bar{\Delta}_{Am}\bar{\Delta}_{Bm} = 0$ and evaluation of the integral results in

$$G(l) = \frac{-k_B T}{8\pi\, l^2}\sum_{n=0}^{\infty}{}^{\dagger}\bar{\Delta}_{Am}\bar{\Delta}_{Bm}\,\overbrace{(1+r_n)\,e^{-r_n}}^{R_n(l,\xi_n)} + \mathcal{O}(\bar{\Delta}_{Am}^2\bar{\Delta}_{Bm}^2)\,, \tag{3}$$

which is a lot more digestible than the original full expression. In this limit, it is clearly seen that for each mode that the sum runs over, there is one clear contribution from the deltas (independent of separation length) and one from $R_n \leq 1$, which follows from retardation as will be pointed out shortly. The imaginary frequency ξ_n, when used in a complex exponent evidently is the rate of decay with time of a charge fluctuation at a certain temperature. Upon revisiting the definition of r_n, it becomes then clear that it is nothing but the ratio of the time it would take light to travel 2 times the distance l in the medium m and the lifetime of a surface mode fluctuation $\frac{1}{\xi_n}$.

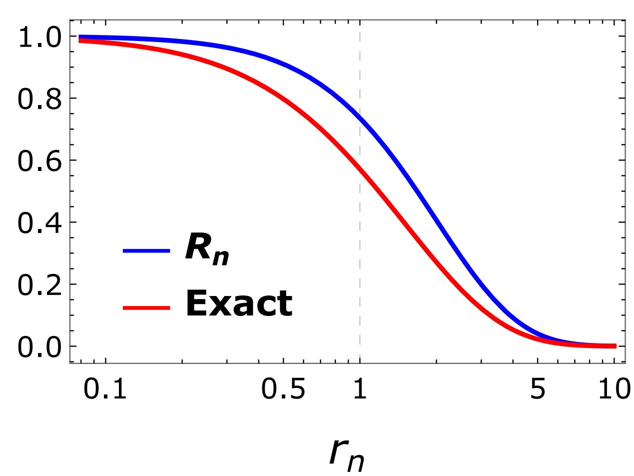


Figure 2: Exact and up to first approximation retardation factor as a function of the ratio of photon travel time over average fluctuation lifetime.

Hence, the case $r_n \gg 1$ corresponds to a scenario where a surface mode fluctuation emits a photon, but dies out before the photon comes back with information about fluctuations at the same frequency on the other side of the medium and therefore, the majority of fluctuations do not contribute to the energy. R_n is a monotonically decreasing function of r_n, with a significant drop in the vicinity of $r_n = 1$, which is in agreement with the intuitive explanation given here (See Figure 2).

The case of two finitely thick (a, b) slabs is the first one to be of a more significant interest because it is the simplest experimentally accessible system. Let the two closest interfaces (the only ones in the infinite slab scenario) be called primary and the next in order carry increasing numbers. As mentioned previously, any pair of interfaces with variable separation contributes to the total force and for two slabs this adds up to four total terms. The original term, arising from the primary interfaces, is clearly dominant in the regime $l \ll a, b$ if, of course separations are small enough for retardation to not play a significant role. It is also very important to point out that sign of the interaction, which determines the attractive or repulsive character of the force, emerges from the relative differences in dielectric constants and magnetic permeabilities ($\bar{\Delta}$, Δ). It is clear from the definition of the relative differences that $\Delta_{ij} = -\Delta_{ji}$. Therefore, the interaction between a primary and a secondary interface will work against the contribution from the two primary interfaces. It follows from here that in the limit where retardation is not significant, the thinner the slabs, the weaker the force due to charge fluctuations.

One might raise the question if there can be repulsive action in a system due to charge fluctuations. In the first approximation limit as in Equation 3, it is evident that a positive $\bar{\Delta}_{Am}\bar{\Delta}_{Bm}$ leads to a negative force per unit area (pressure) and therefore, attractive forces[b]. A direct conclusion is that for two classical materials that are alike, the forces will only be attractive. Intuitively speaking, if the medium m is connected to a large reservoir, as in the Grand canonical ensemble, then the forces between the two slabs become repulsive if it is energetically favourable in terms of interaction of fluctuations for the system to inject more of material m between the slabs. This, in turn, occurs if the interaction $A - m$ is stronger than the interaction of m with itself so that m wants to be in touch with A and leave the reservoir. Additionally, the $B - m$ interaction has to be weaker than any of the other two interactions so that the $A - B$ interaction is weaker and the two slabs prefer m between them rather than sticking to each other. In the aforementioned first approximation, the forces are repulsive if $\epsilon_A < \epsilon_m < \epsilon_B$ at the imaginary frequencies with relevant contribution to the total force[17]. In systems of this sort, it has been found experimentally that such repulsive forces exist[22] (see Figure 3) but only in systems where the wave propagating medium is a liquid. It is crucial to point out that repulsive forces have only been found for long-range interactions where the retardation effect is significant. Additionally, the results and interpretation of the results has been a topic of debate and controversy within the field. In the following section more sophisticated examples of system with repulsive forces due to electromagnetic fluctuations are presented.

III. Repulsive Forces

A repulsive force in a NEMS/MEMS is a very valuable property, which extends the lifetime of the system and protects it from stiction. Already more than twenty years ago, people started questioning the contribution and effect of the Casimir force on the stability of such systems[23,24]. In addition, the rougher a surface, the more likely it is that a driven oscillating plate will exhibit chaotic motion, which leads to stiction and malfunction of the device[25]. It is clear that a repulsive force at small separations will eliminate any points of stable equilibrium at $l = 0$ but a more extensive study is still to be done.

[b] True as long as the energy decays like $l^{-\alpha}$, $\alpha \in \mathbb{R}_{++}$. An increasing force would be unphysical.

Such a study should involve a variety of cases, including ones where the force is both repulsive and attractive at different separations.

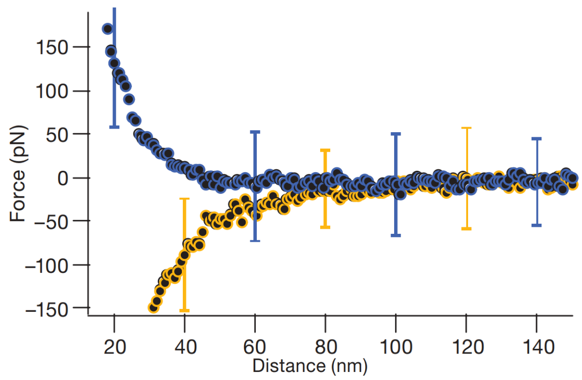


Figure 3: First measurement of repulsive Casimir forces. Blue data represents gold-silica interactions (asymmetric system and repulsive forces) whereas the orange data represents gold-gold interactions (symmetric system and attractive forces). Forces are between a gold sphere and a gold/silica substrate, immersed in bromobenzene. Image is taken from the original paper of Munday *et al*[22]

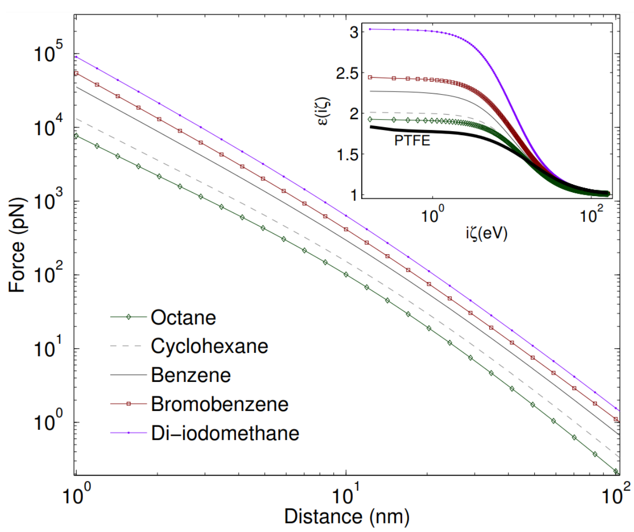


Figure 4: Predicted repulsive Casimir forces and dielectric constants at imaginary frequencies between Polytetrafluoroethylene and gold surfaces for various liquids. Image is taken from the original paper of van Zwol and Palasantzas *et al*[26]

1. Due to geometry and boundary conditions

Even though it may seem like forces due to electromagnetic fluctuations are attractive in nature, as observed in most fluids, it is not so crazy to think about repulsive forces. After all, even vacuum allows for repulsive forces between two bodies but to see how that is possible, one must not ignore the contribution of the magnetic properties. Looking at Equation 1 it is clear that a bigger-than-one argument of the natural logarithm is necessary to derive repulsive forces – a condition, which is satisfied when $\bar{\Delta}_{Am}\bar{\Delta}_{Bm}$, $\Delta_{Am}\Delta_{Bm} < 0$. It can be shown that a general parameters such as $\mu_A = \epsilon_B = 1$, ϵ_A, $\mu_B > 1$ result in Δs with opposite signs in pairs, which in turn satisfies the general condition to have a repulsive force. This result is generalised in the paper of Kenneth *et al*, where repulsive Casimir forces are predicted for a large volume of the parameter space of a system[27]. The approach and the results were criticised one year later and to this date, there is no real conclusion on whether the results should hold or not[28]. In any case, there is no experimental evidence to suggest the claims for repulsive forces to hold. One peculiar and interesting case is repulsion between conducting bodies, solely due to geometry of the setup and its symmetry. The geometry is namely a small elongated body (allowing for electric dipole oscillations) over a perforated plane. The proof involves arguments involving the fluctuation-dissipation theorem[29] and using it to relate the fluctuation energy to the energy of a dipole oscillator. It is shown that the energy when the elongated body is in the centre of the whole is then the same as the energy when the body is at infinity, implying a sign change of the Casimir force at a given point in-between[30]. This is, however, more of a curiosity than an actually applicable result.

The first computed repulsive force is that of a perfect conductor and an infinitely permeable material, computed by Boyer as a sort of pioneering work in the field of repulsive Casimir forces nearly fifty years ago[31]. It was inspired by an even earlier work by Lukosz, which didn't go too deep in the topic of repuslive forces[32]. It should be pointed out that these would be repulsive forces between classical materials[c], arising purely due to boundary conditions at the interfaces, and are not the same as the measured repulsive forces in the paper by Munday *et al*, which is mentioned in the previous section[22]. In general, if one of the boundaries obeys the Neumann boundary condition (continuous first derivative of function) and the other – a Dirichlet boundary condition (continuous function), the force can be repulsive[34]. The issue with repulsive forces from boundary conditions is that it is very hard to tune the magnetic permeability μ of materials and it is also studied significantly less extensively than the dielectric constant. Additionally, repulsive forces in liquids, like the one from the Munday *et al* paper have been shown to exist for a very limited set of systems, and an even smaller subset of systems that exhibit strong repulsive forces[26]. Some of these systems and their predicted repulsive forces are showcased in Figure 4. The limited material choices and the mandatory liquid presence hinder the availability for applications. Fortunately, there are other tricks that one may employ in order to achieve repulsive Casimir forces in vacuum/air, which is what the rest of this paper is dedicated to.

2. Metamaterials

As stated previously, the set of classical materials and geometries, for which a repulsive or tunable Casimir force can be experienced, is extremely limited. This, however, can be changed if metamaterials are used – materials or complex geometries/structures of different composite submaterials, with properties that are tuned for a specific purpose[36] (see Figure 5). The field of metamaterials started with a paper by Veselago, who predicted a metamaterial with a negative dielectric constant and electric permeability in a certain frequency range[37] and to present day such materials have been readily observed[38]. It is imporant to notice that such a metamaterial must be dispersive in order to satisfy conservation of energy[37]. One might be misled that this leads to an unchanged refractive index due to the definition $n = \sqrt{\epsilon\mu}$ but this would only hold if $\epsilon, \mu \in \mathbb{R}$. In fact, for a general optical constants $\epsilon, \mu \in \mathbb{C}$, the following can be stated:

$$\begin{aligned} \text{Let} \quad & \epsilon = r_\epsilon e^{i\phi_\epsilon},\ \mu = r_\mu e^{i\phi_\mu} \\ \implies & n = \sqrt{r_\epsilon r_\mu}\, e^{i\frac{\phi_\epsilon+\phi_\mu}{2}} \\ \implies & \mathrm{Re}[n] < 0 \text{ for } \pi < \phi_\epsilon + \phi_\mu < 2\pi\,. \end{aligned} \tag{4}$$

The imaginary part of the refractive index is related to absorption[d], which is not the focus of the paper. However, a negative real part of the refractive index leads to some interesting effects. For instance, the group velocity of a wave package is opposite to the direction of energy transport (which gives the name *left-handed* to such materials), refraction only reflects a wave, Čerenkov radiation is emitted in a cone behind the charged particle instead of in front, etc.[39] As pointed out by Casimir in the original paper, a material with $\epsilon = \mu = -1$ sitting between the conducting infinite planes will result in a repulsive force, which is not immediately visible from Equation 1. In general, left-handed materials have been used for creation of perfect lenses[40] but their application can also go further. As mentioned previously, left-handed materials can give rise to a repulsive Casimir force, even if not the entire medium between the two slabs is left-handed (useful because such metamaterials are usually solid) as demonstrated first by Leonhardt and Philbin[41]. Despite the appealing result, the model, which is used, is rather primitive and makes some quite crude assumptions regarding the setup and the optical properties. One such assumption is that $\epsilon = \mu = -1$ for all frequencies, which is very unrealistic, and even if the constants were negative for a given frequency range, depending on temperature, if the range is

[c] Classical materials can be defined as homogeneous and infinitely divisible[33].

[d] $\mathrm{Im}[n]$ must be positive so that absorption doesn't amplify the wave instead and energy is conserved. This effectively sets the upper limit of the summed phase to 2π.

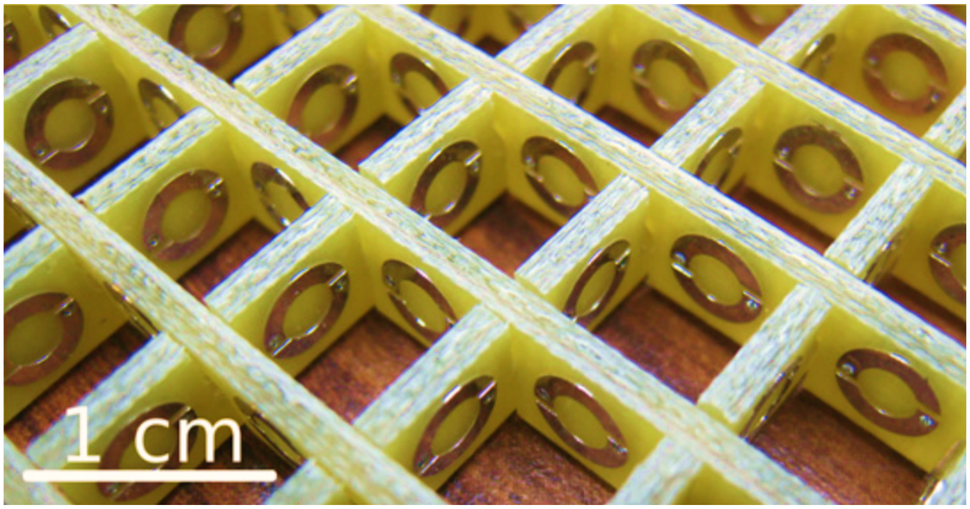


Figure 5: Picture of a chiral metamaterial. Taken from paper by Wang *et al*[35].

small enough and the constants are positive outside, the attraction may outweigh the repulsion.

It turns out that $\epsilon, \mu < 0$ is not a mandatory condition for a negative refractive index and the so-called chiral metamaterials give rise to the same result[42]. In chiral materials the electric field of a propagating wave is coupled to the magnetic field by a coupling parameter $\kappa(\omega)$. Solving the modified Maxwell's equations results in the possibility for the real part of the refractive index to take negative values even if the individual real parts of the dielectric constant and the magnetic permeability are not simultaneously negative at a given frequency range. In particular, the refractive index expression can be modified by adding or subtracting κ from the expression in Equation 4, depending on the direction of circular polarisation of light passing through the material[e]. A pioneering work in the general repulsive force in chiral metamaterials is done by Rosa *et al* but it involves some rather unrealistic modelling of the frequency dependence of the magnetic permeability by a Lorentz-Drude model[43]. Even with its flaws, this paper examines chirality and repulsive forces using a complete Lifshitz theory, unlike the paper by Leonhardt and Philbin, mentioned in the previous paragraph. A follow-up paper by Zhao *et al* offers an improved model for the frequency dependence of the relevant optical parameters. Their conclusion is that a repulsive force is indeed observable, even though it is weaker than that of the pioneering work and only present when the coupling is sufficiently large[44]. This was initially predicted to be true for a couple of known metamaterials[45]. The large, coupling, however, in addition to other restrictions on the parameter space, which make a repulsive force possible, have been shown to be in contradiction with the causality and passivity of a real chiral metamaterial[46]. Additionally, it was later realised that when the theoretical predictions are translated to reality, it is close to impossible to achieve a fully repulsive Casimir force. The negative refractive index does, however, reduce the overall attractive force, which can still be useful for applications in MEMS/NEMS[47]. The usage of chiral metamaterials has also been extended to chiral metamaterial slabs with and without substrate and the theoretical predictions are that repulsive force will be observable[48]. As of present day such a result is yet to be reported by experiment. This is also a potential reason why the seemingly highly active

[e]Any polarisation of light can be expressed as a linear combination of left- and right-handed circular polarisation.

field of Casimir forces and metamaterials around 2010 has been dormant for the past ten years. The exception is made by a brief study of the emerging so-called hyperbolic metamaterials – metamaterials with a heavily anisotropic dielectric tensor, which are of large interest nowadays in the field of nanophotonics[49,50]. Due to modified dispersion relation, it can be shown that hyperbolic metamaterials slabs can also exhibit repulsive Casimir force although once again, there is a lack of experimental evidence[51,52].

Metamaterials are not the only materials where a repulsive force can be obtained in theory. Other examples include graphene[53,54] and arrays of silver nanorods[55,56]. Combined with the highly magnetic materials and coatings, from which the search for repulsive Casimir systems partially started[27], these examples all involve some sort of electronic states in a dimension, lower than three. One particular system, which also exhibits such behaviour are the so-called topological insulators and the next section deals with them exclusively and in more detail.

IV. Topological Insulators

1. What are they?

Imagine a system of one electron, which is confined to two dimensions ($\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$) and is subjected to a constant magnetic field B in the confining dimension $\hat{\mathbf{z}}$. Let the gauge of the electromagnetic potential be such that the scalar potential vanishes and the vector potential is given by $\mathbf{A} = yB\hat{\mathbf{x}}$ so that the (non-relativistic) Hamiltonian of one electron of mass m and charge e is

$$H = \frac{(\mathbf{p} + \frac{e}{c}\mathbf{A})^2}{2m} = \frac{1}{2m}\left((p_x + \tfrac{e}{c}By)^2 + p_y^2,\right) \quad (5)$$

where $\mathbf{p}$ represents momentum in two dimensions. According to quantum mechanics $p_i = -i\hbar\partial_i$ and the energy of a given state E satisfies $H\psi = E\psi$, where ψ is the wavefunction of the electron. The wavefunction can be guessed as a plane wave in x so that $\psi(x, y) = \exp(ik_x x)\,\phi(y)$. Letting the Hamiltonian operator from Equation 5 act on this wavefunction confirms its validity as a solution and results in a harmonic oscillator in y

$$\left(\frac{p_y^2}{2m} - \tfrac{1}{2}m\omega_c^2(y - y_k)^2\right)\phi(y) = E\phi(y);$$
$$\omega_c = \frac{eB}{mc},\; y_k = \frac{eB}{\hbar c}k_x \quad (6)$$
$$\implies E = \hbar\omega_c(n + \tfrac{1}{2}),\; n \in \mathbb{N}$$

Importantly, this is the energy for the electron in both x and y. Even though we said the wavefunction is free in the x direction, which means it can have any momentum, a change of momentum effectively displaces the oscillator in y and does not affect its energy. This makes sense because the 2D system should be invariant to gauge transformations, corresponding to rotations in the $x - y$ plane, and so its energy should not be different in any of the two existing degrees of freedom in that plane. The harmonic oscillator state is highly localised around the potential minimum but as visible by the presence of y_k in Equation 6, the position of the state can be anywhere in the $x - y$ plane. These states can then be represented by parallel lines, concentric circles or vortices, the latter of which is chosen here. In a realistic system, there must be edges present to the plane and these edges start confining states, whose wavefunctions begin to be physically cut-off by the edge. The edge is introduced by a boundary with an insulator. This confinement increases the energy of the states significantly around the edges. Hence, the energy can vary in k-space but only around the edges. So if the system has a particular Fermi energy, which lies between the energies given by the states in Equation 6, then any small electric field perturbation to the system will result in a current but only in the states whose energy has been confined by the edge (see Figure 6).

In the picture where the states are represented by vortices, the edge breaks the vortex and makes it *bounce* on the perimeter of the plane like shown in Figure 6 since the presence of the magnetic field does not allow it to change direction of rotation. The corresponding states are delocalised and conducting. What is described here is an overly-simplified picture of a two-dimensional topological insulator but it gets the message across – the states in the middle are insulating and the states on the edges are conducting. A lot of details are skipped such as having more than one electron, defining the Fermi energy, the role of impurities and their necessity for the definition of a Fermi energy in-between the perfect states, etc.

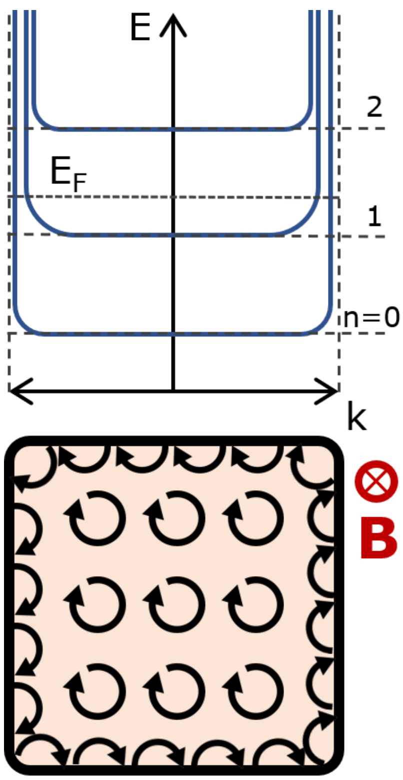


Figure 6: Simplified schematic of the representation of Quantum Hall states by vortices and the corresponding energy-momentum diagram, which shows that the states on the edge can conduct.

All of this is closely related to the Quantum Hall effect, which won't be discussed here, however[59]. Moreover, the conducting states are effectively one-dimensional and propagate a current only in one direction without any back-scattering. A system that has these conducting edges and insulating bulk has a unique so-called *topological invariant* (known as TKNN invariant), which is obtained by a special surface integral over the Brillouin zone[59]. An analogy is the famous topological invariance between a coffee mug and a doughnut (representing a Quantum Hall state), which are topologically different from a ball (representing an insulating state) or some alien doughnut with two holes. In terms of physical symmetry, it can be stated that the magnetic field breaks time-reversal symmetry (a.k.a. T-symmetry) – a necessity for the observation of the effect. The question is whether a topological insulator can exist without the presence of an external field. The first candidate is spin-orbit coupling, which, however, does not come with the appropriate symmetry breaking[60]. It was still used in some pioneering models. The breakthrough came in 2005 when Kane and Mele came up with a realistic model for a topological insulator without an external field and they also showed that a new topological Z_2 invariant exists[f], which also allows conducting edge states in any two-dimensional system as long as it obeys certain conditions[57]. In fact, the invariant corresponds to the number of pairs of conducting edge states. Importantly, a system with Z_2 invariance does not have to break time-reversal symmetry. They showed graphene is one of those materials at sufficiently low temperatures at given ranges of coupling strengths as depicted in Figure 7[57].

A few years later is was realised by a few people almost simultaneously that there are more topological invariants, which would allow conducting surface states in three-dimensional systems[61–63]. Furthermore, it was shown that there are materials, for which these topological invariants exist. More popular and promising ones include Bi_2Te_3, Sb_2Se_3, and Sb_2Te_3[64]. Moreover, the surface states form Dirac cones, which means the surface carriers are in the ultra-relativistic limit $E \propto p$[58] (see 8). In three-dimensional systems it was found that the

[f] Invariant corresponds to a two-fold cyclic group.

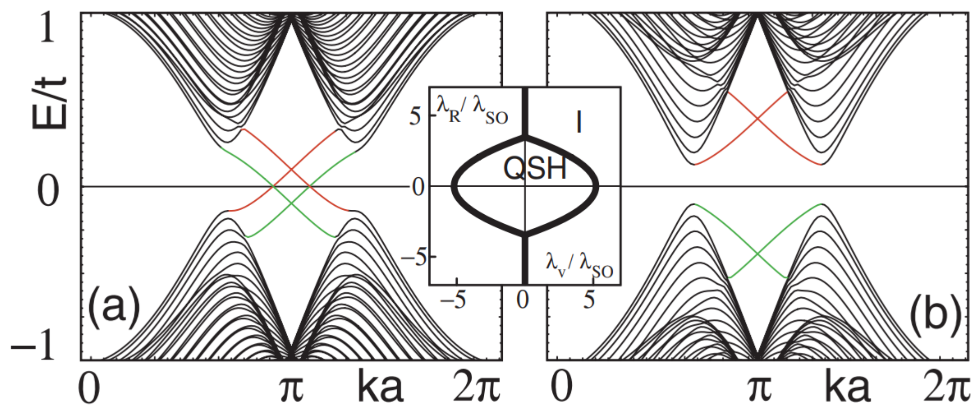


Figure 7: Electronic states energy-momentum diagram for a one-dimensional graphene zigzag strip. On the left there are spin-dependent conducting states and on the right there are none. The conducting states only appear when the staggered lattice potential is sufficiently weak and the parameter space, for which they are observed, is depicted in the central frame. Taken from Kane and Mele's original paper[57]

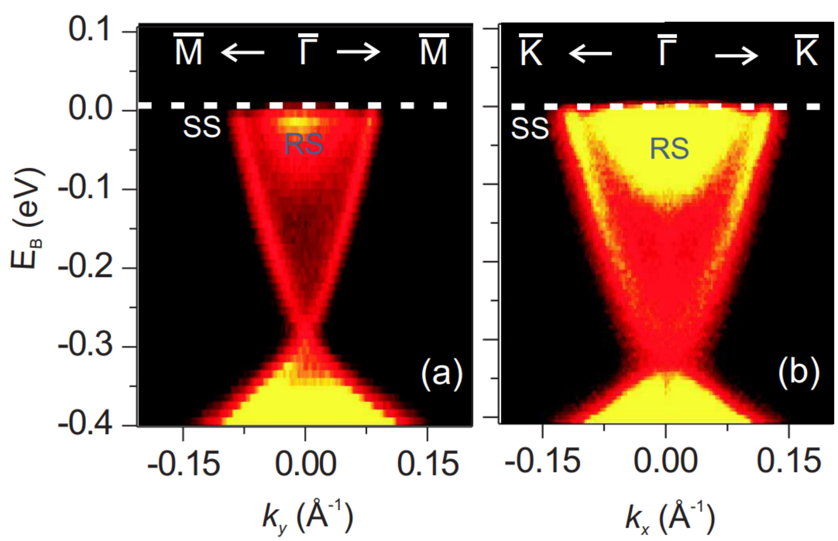


Figure 8: Experimentally measured electronic states of Bi_2Te_3 in the two crystallographic plane directions. The surface states are the thinner lines that go between the bulk states. The Dirac cones in the vicinity of the Γ-point are clearly visible. Taken from the original paper of Xia *et al*[58]

necessary topological invariants can arise from the orbital motion of electrons. This motion introduces a magnetoelectric coupling term in the Lagrangian/Hamiltonian and breaks the time-reversal symmetry, thus, separating the topological insulator (TI) from a normal insulator[65,66]. The modified electrodynamics that follow include a perturbation in the form of a cross-term between the magnetic and electric fields, very similarly to chiral metamaterials, even though the coupling is rooted in a fundamentally different physical phenomenon.

A similar magnetoelectric coupling was known beforehand from Axion[g] physics, which was predicted to have potential in Condensed matter physics decades prior to the discovery of three-dimensional TIs[67]. As it turns out, properties of the Axion particle field can be directly translated to TI physics. This gives a slight taste of already existing Physics into a field, which is full of new ideas and possibilities[68,69]. It should be pointed out that the Axion field coupling does not imply these hypothetical particles exist – it is simply convenient to pretend that the time-reversal symmetry breaking arises from coupling with the Axion field, even though its roots lie elsewhere but the results are identical. It will be shown in the next section how the magnetoelectic coupling comes into play to create repulsive Casimir forces.

Clearly, topological insulators have some very interesting electronic properties and these have made them very promising materials for the future. Some of their theorised or realised applications include quantum computing, superconductors, polarisation splitters, and general spintronics/magnetoelectronics[62,70–74]. The question left to answer now is how are these electronic properties going to impact the Casimir force and also, how are the distinct surface states going to contribute if they are so different from the bulk states.

2. What forces do they give rise to?

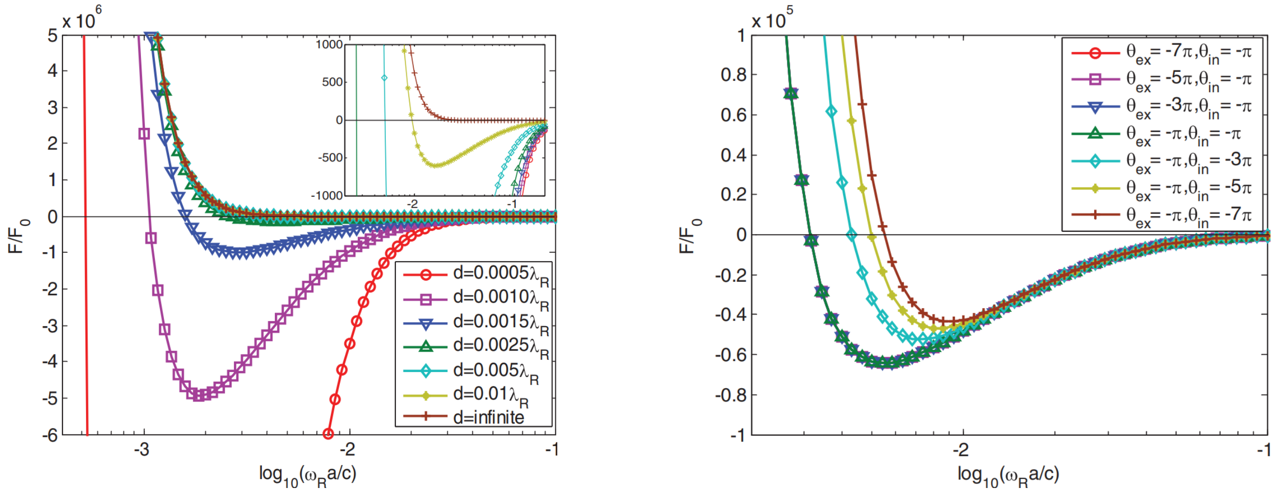
The first challenge for computing a Casimir force on a TI is the approach – how should the surface states be treated? It is in principle possible to introduce the concept of continually varying the dielectric constant and magnetic permeability into the material and compute a Casimir force based on that[17]. It is then a possibility to attach a certain effective *depth* to the surface states and optical constants, which quickly decay to the bulk optical constants away from the surface. This, however, is unfeasible since optical properties are studied with ellipsometry, which is only sensitive to the bulk states, and there are no studies so far with this approach. Another possibility is to treat the electrons as a free two-dimensional gas of Dirac fermions, which turns out to lead to similar results as the aforementioned Axion field coupling[75]. As will be shown, for specific systems of TIs, the Casimir force can be tuned to become repulsive.

The first approach to be introduced is one utilising an Axion field coupling and was done by Grushin and Cortijo in 2011, amongst the Casimir repulsion craze in the field[77]. The Lagrangian density of the system can be written as

$$\mathcal{L} = \underbrace{\epsilon \mathbf{E}^2 - \frac{1}{\mu}\mathbf{B}^2}_{\text{Classical}} + \theta \frac{\alpha}{4\pi^2}\mathbf{E}\cdot\mathbf{B}\,, \tag{7}$$

where the last term represents the coupling to the Axion field by a parameter θ, also known as the Topological magnetoelectric polarisablity. $\alpha \approx \frac{1}{137}$ is the fine structure constant. Since $\mathbf{B}$ breaks T-symmetry and $\mathbf{E}$ does not, their inner product breaks T-symmetry and therefore also the newly added term to the Lagrangian density breaks it. Hence, $\theta = 0$ corresponds to a normal insulator and TI bulk states are identified by $\theta = \pi$. Upon introduction of an additional external T-symmetry breaking perturbation on the surface, θ can be modified to take values $\theta = (2n+1)\pi$, $n \in \mathbb{Z}$. The exact value of θ then depends on the nature of the

[g] Axions are hypothetical elementary particles, proposed to solve the so-called *strong CP problem* in Quantum chromodynamics.



(a) (b)

Figure 9: Casimir force per unit area (in units of $\hbar\omega_R^4/(c^3 2\pi^2)$) versus a dimensionless slab separation parameter on a logarithmic scale. ω_R is the resonance frequency of the pure oscillator model, used for computation of the force. In **(a)**, computation at different TI slab thicknesses are presented whereas in **(b)**, the computation is performed for varying magnetoelectric coupling strengths. θ_{in} corresponds to the coatings on the side of the gap between the TI slabs and θ_{ex} – to the other two coatings. Taken from the original paper by Nie *et al*[76].

perturbation. Such perturbations can be realised experimentally the easiest by an application of a thin magnetised ferromagnetic layer. In Grushin and Cortijo's paper, the dielectric response of the bulk is modelled by a Lorentz-Drude oscillator, which is arguably a good approximation for topologically insulating materials such as $TlBeSe_2$[78]. Within the given approximations, it can then be shown that for a system of two semi-infinite TIs (as in Figure 1) in vacuum with magnetic coatings facing towards the gap, corresponding to θ_1 and θ_2, a repulsive Casimir force exists at smaller separations if θ_1 and θ_2 have opposite signs (opposite magnetisations with respect to the surface normal vectors). The force is also weakly attractive at larger separations and a point of zero force can be predicted. The repulsive force is the largest for $\theta_1 = -\theta_2 = \pi$[77]. Although the result looks appealing, there are some simplifications and approximations that might prevent an experimental realisation such as the aforementioned pure oscillator model for the dielectric response. Another one is the assumption that the Axion coupling Lagrangian density is a good description for an infinitely large frequency range[69]. Nonetheless, the general result within the approximations is that the Casimir force can be tuned and switched from repulsive to attractive by the application of suitable magnetic layers on the TI surface. The result was built upon by Chen and Wan who show a generalised picture with its respective limits[75]. Later, Martinez and Jalil showed that the forces can be further tuned by introduction of free surface charges, which effectively play the same role as the Axion field coupling[79]. Finally, within the same formalism and pool of assumptions, Nie *et al* showed that the results can be extended to systems with TI slabs (four θ-associated magnetic coatings in total) with and without substrate[76]. Moreover, the presence of a substrate (Silicon) strengthens the attractive forces and pushes the zero-force point towards smaller separations. Most notably, the study shows that the slab separation at the zero-force point decreases with decreasing slab thickness but thinner slabs also exhibit stronger attractive forces (see Figure 9a, 9b). The stronger attractive forces for thinner slabs is not in agreement with the result for two classical slabs, which is discussed in the first section. The thickness-dependence can be complemented with a study by Post *et al* where it is shown that thin TI films can effectively modify the parameters in an oscillator model and exhibit smaller-than-expected band gaps[80]. A study, which implements these findings into Casimir force calculations is yet to be performed. Finally, an increasing dielectric constant of the TI leads to a decreasing zero-force separation and the temperature of the system also plays a significant role, which is, however, nontrivial.

What is left to be done in the field is to bring these topological insulator models to reality. The range and magnitude of the Casimir forces in TIs are large enough to be accessible experimentally by atomic force microscopy (AFM) techniques or Fabry-Perot type cavities[76]. There are, of course, quite a few complications to be encountered in that direction. First of all, a measurement of the dielectric constant in a film is necessary for computation of the forces from optical data, which can then be compared to actual force measurements. The optical measurements have to be performed, however, over an extremely wide range of frequencies, which is rather challenging by itself. Furthermore, any optical measurement will pick up information about the dielectric response in the bulk but not the surface. This leads us to the next point – there is a lack of realistic studies on the contribution of surface states to the Casimir force. Modelling the system by continuously varying the dielectric constant around the surface could lead to a more detailed understanding. One significant challenge with experimental measurement of the Casimir force by AFM is the presence of residual charge on the surfaces, which has to be measured beforehand and its contribution has to be subtracted from the measurement.

Finally, for finding applications of a repulsive Casimir force, there have to be studies, which delve deeper into the dynamics and stability of systems with repulsive Casimir forces. Such studies are already present for regular Casimir systems with attractive forces and are of extreme importance in NEMS/MEMS[25]. Studies regarding the dynamics of repulsive Casimir systems will be necessary to point out the superiority of such systems in electromechanical systems but perhaps are only logical to appear after sufficient experimental understanding of the repulsive Casimir forces is reached. Thus, the first priority for topological insulators and any other materials/systems, for which repulsive forces have been theorised, is to observe and understand the repulsive interaction by means of experiment, including the contribution of surface states via an appropriate model.

V. Conclusion

Understanding the intermolecular and interatomic forces has been of great interest since the development of thermodynamics in the late 19$^{\text{th}}$ century and is just as relevant today in fields ranging from Theoretical physics to Medicine. The most important contribution to the understanding of these forces was made by Lifshitz in the mid-20$^{\text{th}}$ century who showed that as long as the optical response of a system in a wide enough frequency range is known, which is a challenge by itself, the forces due to fluctuations in the electromagnetic field can be computed. One realisation that followed was that under specific conditions the Lifshitz theory predicts that the forces could turn repulsive and this opens an ocean of application possibilities ranging from quantum levitation to nano/micro electromechanical systems without stiction and only imagination can be the limit if the Casimir forces are fully tamed[81]. In liquid, repulsive forces have been predicted but the validity of their subsequent measurement is disputed. In air and vacuum it becomes clear that the overwhelming majority of classical materials will result in attractive forces, with the exception being asymmetric systems involving materials with strong magnetic response but even this result is disputed and not set-in-stone. Systems that hold the greatest promise are built from a new generation of materials – metamaterials and topological insulators. Coincidentally, in both the former and the latter, a degree of coupling between the electric and magnetic field within the material is present, which modifies the underlying electrodynamics and is theorised within a set of approximations to result in repulsive forces. Three-dimensional topological insulators appear more promising due to being easier to work with than metamaterials and offering a wider range of tunability in the force. To this day, however, there is a lack of experimental evidence to support any of the theoretical predictions and a lot of work is still to be done. Some of the challenges that remain to be tackled include an understanding of the contribution of surface states in topological insulators, their implementation in experiment and dependence on sample thickness, presence of a substrate, temperature, etc. Tackling some of these challenges might revive the field of repulsive Casimir forces and unlock the gate towards application but also, fundamental understanding of both the materials involved and the fluctuations of the electromagnetic field, related to the problem of zero-point energy, as well as how the different elements of the problem intertwine.

References

[1] J. D. Van der Waals, *Over de Continuiteit van den Gas-en Vloeistoftoestand*. Sijthoff, 1873, vol. 1.

[2] B. V. Deryagin, "P. n. LEBEDEV's IDEAS ON THE NATURE OF MOLECULAR FORCES," *Soviet Physics Uspekhi*, vol. 10, no. 1, pp. 108–111, jan 1967. [Online]. Available: https://doi.org/10.1070/pu1967v010n01abeh003203

[3] R. Eisenschitz and F. London, "Über das verhältnis der van der waalsschen kräfte zu den homöopolaren bindungskräften," *Zeitschrift für Physik*, vol. 60, no. 7-8, pp. 491–527, 1930.

[4] F. London, "The general theory of molecular forces," *Transactions of the Faraday Society*, vol. 33, pp. 8b–26, 1937.

[5] J. Roberts and W. Orr, "Induced dipoles and the heat of adsorption of argon on ionic crystals," *Transactions of the Faraday Society*, vol. 34, pp. 1346–1349, 1938.

[6] W. Keesom, "The second viral coefficient for rigid spherical molecules, whose mutual attraction is equivalent to that of a quadruplet placed at their centre," *Proc. R. Acad. Sci*, vol. 18, pp. 636–646, 1915.

[7] D. J. Griffiths, "Introduction to electrodynamics," 2005.

[8] D. J. Griffiths and D. F. Schroeter, *Introduction to quantum mechanics*. Cambridge University Press, 2018.

[9] H. B. Casimir, "On the attraction between two perfectly conducting plates," in *Proc. Kon. Ned. Akad. Wet.*, vol. 51, 1948, p. 793.

[10] H. B. Casimir and D. Polder, "The influence of retardation on the london-van der waals forces," *Physical Review*, vol. 73, no. 4, p. 360, 1948.

[11] E. LIFSHITZ and M. Hamermesh, "The theory of molecular attractive forces between solids," in *Perspectives in Theoretical Physics*. Elsevier, jan 1992, pp. 329–349.

[12] I. E. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitaevskii, "The general theory of van der waals forces," *Advances in Physics*, vol. 10, no. 38, pp. 165–209, 1961.

[13] N. Van Kampen, B. Nijboer, and K. Schram, "On the macroscopic theory of van der waals forces," *Physics letters A*, vol. 26, no. 7, pp. 307–308, 1968.

[14] D. Langbein, "Theory of van der waals attraction," in *Springer tracts in modern physics*. Springer, 1974, pp. 1–139.

[15] J. Mahanty and B. W. Ninham, *Dispersion forces*. Academic Press, 1976, vol. 1.

[16] Y. S. Barash and V. L. Ginzburg, "Electromagnetic fluctuations in matter and molecular (van-der-waals) forces between them," *Soviet Physics Uspekhi*, vol. 18, no. 5, p. 305, 1975.

[17] V. Adrian Parsegian, "Van der Waals forces: A handbook for biologists, chemists, engineers, and physicists," pp. 1–380, 2005. [Online]. Available: https://books.google.nl/books?hl=en&lr=&id=K4F7uHJVEOoC&oi=fnd&pg=PP1&dq=parsegian&ots=HUEikwRwzT&sig=5TyOV0MhxLcqO5XreB0idwXrCJY&redir_esc=y#v=onepage&q=parsegian&f=false

[18] L. D. Landau, J. Bell, M. Kearsley, L. Pitaevskii, E. Lifshitz, and J. Sykes, *Electrodynamics of continuous media*. Elsevier, 1984, vol. 8.

[19] G. Klimchitskaya, A. Roy, U. Mohideen, and V. Mostepanenko, "Complete roughness and conductivity corrections for casimir force measurement," *Physical Review A*, vol. 60, no. 5, p. 3487, 1999.

[20] P. Van Zwol, G. Palasantzas, and J. T. M. De Hosson, "Influence of random roughness on the casimir force at small separations," *Physical Review B*, vol. 77, no. 7, p. 075412, 2008.

[21] P. M. Neto, A. Lambrecht, and S. Reynaud, "Roughness correction to the casimir force: Beyond the proximity force approximation," *EPL (Europhysics Letters)*, vol. 69, no. 6, p. 924, 2005.

[22] J. N. Munday, F. Capasso, and V. A. Parsegian, "Measured long-range repulsive Casimir-Lifshitz forces," *Nature*, vol. 457, no. 7226, pp. 170–173, jan 2009. [Online]. Available: /pmc/articles/PMC4169270//pmc/articles/PMC4169270/?report=abstracthttps://www.ncbi.nlm.nih.gov/pmc/articles/PMC4169270/

[23] F. M. Serry, D. Walliser, and G. J. Maclay, "The anharmonic casimir oscillator (aco)-the casimir effect in a model microelectromechanical system," *Journal of Microelectromechanical Systems*, vol. 4, no. 4, pp. 193–205, 1995.

[24] E. Buks and M. L. Roukes, "Metastability and the casimir effect in micromechanical systems," *EPL (Europhysics Letters)*, vol. 54, no. 2, p. 220, 2001.

[25] W. Broer, H. Waalkens, V. B. Svetovoy, J. Knoester, and G. Palasantzas, "Nonlinear Actuation Dynamics of Driven Casimir Oscillators with Rough Surfaces," *Physical Review Applied*, vol. 4, no. 5, p. 054016, nov 2015. [Online]. Available: https://journals.aps.org/prapplied/abstract/10.1103/PhysRevApplied.4.054016

[26] P. Van Zwol and G. Palasantzas, "Repulsive casimir forces between solid materials with high-refractive-index intervening liquids," *Physical Review A*, vol. 81, no. 6, p. 062502, 2010.
[27] O. Kenneth, I. Klich, A. Mann, and M. Revzen, "Repulsive Casimir Forces," *Physical Review Letters*, vol. 89, no. 3, p. 033001, jun 2002. [Online]. Available: https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.89.033001
[28] D. Iannuzzi and F. Capasso, "Comment on Repulsive Casimir Forces," p. 029101, jul 2003. [Online]. Available: https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.91.029101
[29] H. B. Callen and T. A. Welton, "Irreversibility and generalized noise," *Physical Review*, vol. 83, no. 1, p. 34, 1951.
[30] M. Levin, A. P. McCauley, A. W. Rodriguez, M. H. Reid, and S. G. Johnson, "Casimir repulsion between metallic objects in vacuum," *Physical review letters*, vol. 105, no. 9, p. 090403, 2010.
[31] T. H. Boyer, "Van der waals forces and zero-point energy for dielectric and permeable materials," *Physical Review A*, vol. 9, no. 5, p. 2078, 1974.
[32] W. Lukosz, "Electromagnetic zero-point energy and radiation pressure for a rectangular cavity," *Physica*, vol. 56, no. 1, pp. 109–120, 1971.
[33] G. Carcassi, C. A. Aidala, D. J. Baker, and L. Bieri, "From physical assumptions to classical and quantum hamiltonian and lagrangian particle mechanics," *Journal of Physics Communications*, vol. 2, no. 4, p. 045026, 2018.
[34] T. H. Boyer, "Casimir forces and boundary conditions in one dimension: Attraction, repulsion, planck spectrum, and entropy," *American Journal of Physics*, vol. 71, no. 10, pp. 990–998, 2003.
[35] B. Wang, J. Zhou, T. Koschny, and C. M. Soukoulis, "Nonplanar chiral metamaterials with negative index," *Applied Physics Letters*, vol. 94, no. 15, p. 151112, 2009.
[36] R. S. Kshetrimayum, "A brief intro to metamaterials," *IEEE Potentials*, vol. 23, no. 5, pp. 44–46, 2004.
[37] V. G. Veselago, "The electrodynamics of substances with simultaneously negative values of ϵ and μ," *Physics-Uspekhi*, vol. 10, no. 4, pp. 509–514, 1968.
[38] R. A. Shelby, D. R. Smith, and S. Schultz, "Experimental verification of a negative index of refraction," *science*, vol. 292, no. 5514, pp. 77–79, 2001.
[39] P. W. Milonni, *Fast light, slow light and left-handed light*. CRC Press, 2004.
[40] J. B. Pendry, "Negative refraction makes a perfect lens," *Physical review letters*, vol. 85, no. 18, p. 3966, 2000.
[41] U. Leonhardt and T. G. Philbin, "Quantum levitation by left-handed metamaterials," *New Journal of Physics*, vol. 9, no. 8, p. 254, 2007.
[42] B. Wang, J. Zhou, T. Koschny, M. Kafesaki, and C. M. Soukoulis, "Chiral metamaterials: simulations and experiments," *Journal of Optics A: Pure and Applied Optics*, vol. 11, no. 11, p. 114003, 2009.
[43] F. S. Rosa, D. A. Dalvit, and P. W. Milonni, "Casimir-lifshitz theory and metamaterials," *Physical Review Letters*, vol. 100, no. 18, p. 183602, may 2008. [Online]. Available: https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.100.183602
[44] R. Zhao, J. Zhou, T. Koschny, E. Economou, and C. Soukoulis, "Repulsive casimir force in chiral metamaterials," *Physical review letters*, vol. 103, no. 10, p. 103602, 2009.
[45] R. Zhao, T. Koschny, E. Economou, and C. Soukoulis, "Comparison of chiral metamaterial designs for repulsive casimir force," *Physical Review B*, vol. 81, no. 23, p. 235126, 2010.
[46] M. G. Silveirinha and S. I. Maslovski, "Comment on "repulsive casimir force in chiral metamaterials"," *Physical review letters*, vol. 105, no. 18, p. 189301, 2010.
[47] R. Zhao, T. Koschny, E. Economou, and C. Soukoulis, "Chiral metamaterials reduce the attractive casimir force," in *Metamaterials: Fundamentals and Applications III*, vol. 7754. International Society for Optics and Photonics, 2010, p. 77540Z.
[48] ——, "Repulsive casimir forces with finite-thickness slabs," *Physical Review B*, vol. 83, no. 7, p. 075108, 2011.
[49] A. Poddubny, I. Iorsh, P. Belov, and Y. Kivshar, "Hyperbolic metamaterials," *Nature photonics*, vol. 7, no. 12, pp. 948–957, 2013.
[50] L. Ferrari, C. Wu, D. Lepage, X. Zhang, and Z. Liu, "Hyperbolic metamaterials and their applications," *Progress in Quantum Electronics*, vol. 40, pp. 1–40, 2015.
[51] G. Song, J. Xu, C. Zhu, P. He, Y. Yang, and S.-Y. Zhu, "Casimir force between hyperbolic metamaterials," *Physical Review A*, vol. 95, no. 2, p. 023814, 2017.
[52] G. Song, R. Zeng, M. Al-Amri, J. Xu, C. Zhu, P. He, and Y. Yang, "Repulsive casimir force between hyperbolic metamaterials," *Optics express*, vol. 26, no. 26, pp. 34 461–34 473, 2018.
[53] J. Martinez and M. Jalil, "Casimir force between metal and graphene sheets," *JOSA B*, vol. 32, no. 1, pp. 157–162, 2015.
[54] J. Martinez, X. Chen, and M. Jalil, "Casimir effect and graphene: Tunability, scalability, casimir rotor," *AIP Advances*, vol. 8, no. 1, p. 015330, 2018.
[55] S. I. Maslovski and M. G. Silveirinha, "Ultralong-range casimir-lifshitz forces mediated by nanowire materials," *Physical Review A*, vol. 82, no. 2, p. 022511, 2010.
[56] ——, "Mimicking boyer's casimir repulsion with a nanowire material," *Physical Review A*, vol. 83, no. 2, p. 022508, 2011.
[57] C. L. Kane and E. J. Mele, "Z2 topological order and the quantum spin hall effect," *Physical Review Letters*, vol. 95, no. 14, sep 2005.
[58] Y. Xia, L. Wray, D. Qian, D. Hsieh, A. Pal, H. Lin, A. Bansil, D. Grauer, Y. Hor, R. Cava *et al.*, "Electrons on the surface of bi2se3 form a topologically-ordered two dimensional gas with a non-trivial berry's phase," *arXiv preprint arXiv:0812.2078*, 2008.
[59] K. v. Klitzing and G. Ebert, *The quantum hall effect*, 2nd ed. Springer Science Business Media, 2012, 1983, vol. 117-118, no. PART 2. [Online]. Available: https://books.google.nl/books?hl=en&lr=&id=mxrSBwAAQBAJ&oi=fnd&pg=PR5&dq=The+Quantum+Hall+Effect&ots=4vfqnTH1m1&sig=TH9qz3eQ97MJHql5mrxc5jEyb38&redir_esc=y#v=onepage&q=TheQuantumHallEffect&f=false
[60] J. E. Moore, "The birth of topological insulators," *Nature*, vol. 464, no. 7286, pp. 194–198, mar 2010. [Online]. Available: https://www.semanticscholar.org/paper/The-birth-of-topological-insulators-Moore/4a252e154263c87ff1341c44c510ac41fc62b27a
[61] R. Roy, "Three dimensional topological invariants for time reversal invariant Hamiltonians and the three dimensional quantum spin Hall effect," *Physical Review B - Condensed Matter and Materials Physics*, vol. 79, no. 19, jul 2006. [Online]. Available: http://arxiv.org/abs/cond-mat/0607531http://dx.doi.org/10.1103/PhysRevB.79.195322
[62] J. E. Moore and L. Balents, "Topological invariants of time-reversal-invariant band structures," *Physical Review B - Condensed Matter and Materials Physics*, vol. 75, no. 12, p. 121306, mar 2007. [Online]. Available: https://journals.aps.org/prb/abstract/10.1103/PhysRevB.75.121306
[63] L. Fu, C. L. Kane, and E. J. Mele, "Topological insulators in three dimensions," *Physical Review Letters*, vol. 98, no. 10, p. 106803, mar 2007. [Online]. Available: https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.98.106803
[64] H. Zhang, C. X. Liu, X. L. Qi, X. Dai, Z. Fang, and S. C. Zhang, "Topological insulators in Bi 2 Se 3, Bi 2 Te 3 and Sb 2 Te 3 with

a single Dirac cone on the surface," *Nature Physics*, vol. 5, no. 6, pp. 438–442, jun 2009.

[65] M. Fiebig, "Revival of the magnetoelectric effect," *Journal of physics D: applied physics*, vol. 38, no. 8, p. R123, 2005.

[66] F. W. Hehl, Y. N. Obukhov, J.-P. Rivera, and H. Schmid, "Relativistic analysis of magnetoelectric crystals: Extracting a new 4-dimensional p odd and t odd pseudoscalar from cr2o3 data," *Physics Letters A*, vol. 372, no. 8, pp. 1141–1146, 2008.

[67] F. Wilczek, "Two applications of axion electrodynamics," *Physical Review Letters*, vol. 58, no. 18, pp. 1799–1802, may 1987. [Online]. Available: https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.58.1799

[68] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, "Topological field theory of time-reversal invariant insulators," *Physical Review B*, vol. 78, no. 19, p. 195424, 2008.

[69] A. M. Essin, J. E. Moore, and D. Vanderbilt, "Magnetoelectric polarizability and axion electrodynamics in crystalline insulators," *Physical review letters*, vol. 102, no. 14, p. 146805, 2009.

[70] L. Fu and C. L. Kane, "Superconducting proximity effect and majorana fermions at the surface of a topological insulator," *Physical Review Letters*, vol. 100, no. 9, mar 2008.

[71] M. Z. Hasan and C. L. Kane, "Colloquium: Topological insulators," *Disciplines Physical Sciences and Mathematics | Physics Comments Suggested Citation*, vol. 82, pp. 3045–3067, 2010. [Online]. Available: http://repository.upenn.edu/physics_papershttp://dx.doi.org/10.1103/RevModPhys.82.3045.ThisjournalarticleisavailableatScholarlyCommons:http://repository.upenn.edu/physics_papers/39

[72] F. Liu, J. Xu, and Y. Yang, "Polarization conversion of reflected electromagnetic wave from topological insulator," *JOSA B*, vol. 31, no. 4, pp. 735–741, 2014.

[73] X. L. Qi and S. C. Zhang, "Topological insulators and superconductors," *Reviews of Modern Physics*, vol. 83, no. 4, p. 1057, oct 2011. [Online]. Available: https://journals.aps.org/rmp/abstract/10.1103/RevModPhys.83.1057

[74] J. Yin, H. N. Krishnamoorthy, G. Adamo, A. M. Dubrovkin, Y. D. Chong, N. I. Zheludev, and C. Soci, "Plasmonics of topological insulators at optical frequencies," p. 425, feb 2017. [Online]. Available: www.nature.com/am

[75] L. Chen and S. Wan, "Casimir interaction between topological insulators with finite surface band gap," *Physical Review B - Condensed Matter and Materials Physics*, vol. 84, no. 7, p. 075149, aug 2011. [Online]. Available: https://journals.aps.org/prb/abstract/10.1103/PhysRevB.84.075149

[76] W. Nie, R. Zeng, Y. Lan, and S. Zhu, "Casimir force between topological insulator slabs," *Physical Review B - Condensed Matter and Materials Physics*, vol. 88, no. 8, p. 085421, aug 2013. [Online]. Available: https://journals.aps.org/prb/abstract/10.1103/PhysRevB.88.085421

[77] A. G. Grushin and A. Cortijo, "Tunable casimir repulsion with three-dimensional topological insulators," *Physical Review Letters*, vol. 106, no. 2, p. 020403, jan 2011. [Online]. Available: https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.106.020403

[78] T. Sato, K. Segawa, H. Guo, K. Sugawara, S. Souma, T. Takahashi, and Y. Ando, "Direct evidence for the dirac-cone topological surface states in the ternary chalcogenide tlbise 2," *Physical review letters*, vol. 105, no. 13, p. 136802, 2010.

[79] J. C. Martinez and M. B. Jalil, "Tuning the Casimir force via modification of interface properties of three-dimensional topological insulators," *Journal of Applied Physics*, vol. 113, no. 20, p. 204302, may 2013. [Online]. Available: http://aip.scitation.org/doi/10.1063/1.4807796

[80] K. Post, B. Chapler, L. He, X. Kou, K. L. Wang, and D. Basov, "Thickness-dependent bulk electronic properties in bi 2 se 3 thin films revealed by infrared spectroscopy," *Physical Review B*, vol. 88, no. 7, p. 075121, 2013.

[81] P. Ball, "Fundamental physics: Feel the force," 2007.