Exules

Hammad Rana

Abstract: This paper will be dedicated to establishing the concept of exules. The term exul (singular for exules) means exile in Latin. So in a sense an exul is essentially the exiled data from a geometric structure that is not discovered from conventional (differential or algebraic) geometric tools. We will use categories and model theory as our main tools to inspect exules. Exules of polynomials will serve as the main and initial concept we are interested in exploring but we will also go over exules of manifolds and other kinds of structures. The final talk will be on generalizing the concept of exules to work with any mathematical structure.

1 Exules over Polynomials

1.1 Equation of a circle

Suppose we have a multivariate polynomial function that represents the equation of the circle, centered at (0,0).

$$Z = X^2 + Y^2 = R^2 \tag{1}$$

We have a commutative diagram that represents the relations.

$$\begin{array}{ccc} X & \xrightarrow{Z} & Y \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle R} \\ V & \xrightarrow{1/R} & W \end{array}$$

The upper part of the commutative diagram represents that Z is a function that relates X and Y, which are sets of the coordinates (x, y) respectively. The right side of the diagram shows that the set of Y-coordinates is related with W, the set of all solutions, via the the function of the radius R. On the bottom, we see that there exists an inverse function from V, the projective algebraic variety (which itself is a set of solutions to the polynomial but embedded within $\mathbf{P}^n$), and W. Finally, the left side is that there is a function that represents the "bounds" of the equation and it relates to the equation with the variety. The bounds are the initial solutions one can come up quickly with basic algebra without any respect to a fixed or given constant value. First we can write an alternative form for the equation:

$$-R^2 + X^2 + Y^2 = 0 \tag{2}$$

The bounds are then

$$\begin{aligned} R &= \sqrt{X^2 + Y^2} \\ X &= \sqrt{R^2 - Y^2} \\ Y &= \sqrt{R^2 - X^2} \end{aligned} \tag{3}$$

The function of the equation is in a sense "bounded" by these solutions as they complete its equation. We have a function of the bounds β and it relates the equation with the algebraic variety. Since the entire diagram commutes we can write the following compositions:

$$\begin{aligned} Z &: X \circ Y \\ Z \circ R &: X \circ W \end{aligned} \tag{4}$$

For this diagram to commute we have:

$$R \circ Z = 1/R \circ \beta \tag{5}$$

The category of all sets here can be denoted C. So now for any set in C we allow for an interpretation function $\mathcal{I}$ to exist alongside any such set in an ordered pair. We choose the set of all solutions W and we do so since it contains all solutions to the equation and would be good enough to provide an example for an exul. The algebraic variety will serve its purpose at the end. Anyhow, we denote the ordered pair $(W, \mathcal{I})$. This ordered pair is one such for a language $\mathcal{L}$ which denotes the set of all of logical symbols and in our case will contain the functions and relations from $Z \circ R$. We have the logical symbols $\wedge, \vee, \neg, \implies, \iff$ called "and", "or", "not", "implies" and "iff" respectively, the quantifiers $\forall$ and $\exists$ called "for all" and "there exists," an infinite collection of natural variables, two parentheses), (, and the equal sign symbol =. The symbols for our functions are (Z, R) and the symbols for our relations are the arrows. The arrows can be denoted $\mathcal{R}_2$, the particular subscript since there are two arrows in question. Our ordered pair is equivalent to our model $\mathfrak{M}$. Allow for W to be noted as the universe of the model, and the model itself is both the universe W and the interpretation function $\mathcal{I}$. The model in a sense can act like both. We write two triples: $(Z, X, Y), (R, Y, W)$, meaning Z is a function relating X and Y and R is a function relating Y and W respectively. Using the second statement in (4) we have that R relates both X and Y to W as well.

Now let us take a look at W. First off, the equation of the circle in (1) is the equation of a curve that bounds a circle, and allows us to determine the circumference of one. We switch it over to an inequality.

$$X^2 + Y^2 \leq R^2 \tag{6}$$

Then we have

$$\begin{aligned} X &= \pm\sqrt{R^2} \\ Y &= 0 \\ -\sqrt{R^2} < X &< \sqrt{R^2} \\ -\sqrt{R^2 - X^2} \leq Y &\leq \sqrt{R^2 - X^2} \\ [-\sqrt{R^2 - X^2}, \sqrt{R^2 - X^2}], R &\in \mathbb{R} \\ (-|R|, |R|) \end{aligned} \tag{7}$$

All the values of X and Y above being elements of W.