MATH 446 - May 04/11

(1) $J = \begin{pmatrix} 0 & E_m \\ -E_m & 0 \end{pmatrix}$, $Sp(m,\mathbb{R}) = \{X \in Mat_{2m}(\mathbb{R}) \mid X^t J X = J\}$

a) Take $X, Y \in Sp(m,\mathbb{R})$. Clearly $\det X^t J X = \det X^t \det J \det X = 1$. Since $\det J \neq 0$, then $\det X \neq 0$ or the statement would make no sense. $X, Y \in Sp(m,\mathbb{R})$ gives $(XY)^t J XY = Y^t X^t J X Y$

$= Y^t J Y = J \Rightarrow XY \in Sp(m,\mathbb{R})$.

$X \in Sp(m,\mathbb{R}) \Rightarrow X^t J X = J \Rightarrow J = (X^t)^{-1} J X = (X^{-1})^t J X$

$\Rightarrow X^{-1} \in Sp(m,\mathbb{R})$. Also, $X = E \Rightarrow EJE = J$ is satisfied, so $E \in Sp(m,\mathbb{R})$. Thus $Sp(m,\mathbb{R}) \subset GL(2m,\mathbb{R})$ is a subgroup.

$\Rightarrow Y = (E - X)(E+X)^{-1}$, defined for X near E since $\det(E+E) \neq 0$.

b) $X = (E-Y)(E+Y)^{-1}$, $Y \in Mat_{2m}(\mathbb{R})$, Y near 0.

$X^t J X = J$ gives $[(E-Y)(E+Y)^{-1}]^t J (E-Y)(E+Y)^{-1} = J$

$\Rightarrow (E + Y^t)^{-1}(E - Y^t) J (E-Y)(E+Y)^{-1} = J$

$\Rightarrow (E - Y^t) J (E - Y) = (E + Y^t) J (E + Y)$

$\Rightarrow (J - Y^t J)(E - Y) = (J + Y^t J)(E + Y) \Rightarrow \cancel{J} + \cancel{Y^t J Y} - Y^t J - JY = \cancel{J} + \cancel{Y^t J Y} + Y^t J + JY$

$\Rightarrow 2Y^t J = -2JY \Rightarrow Y^t J = -JY$.

c) Above gives a linear condition on Y for X to be in $Sp(m,\mathbb{R})$.

$Y^t J = -JY \Rightarrow$ if we write Y blockwise as $Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$,

we have $\begin{pmatrix} y_{11}^t & y_{21}^t \\ y_{12}^t & y_{22}^t \end{pmatrix}\begin{pmatrix} 0 & E_m \\ -E_m & 0 \end{pmatrix} = \begin{pmatrix} 0 & -E_m \\ E_m & 0 \end{pmatrix}\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -y_{21}^t & y_{11}^t \\ -y_{22}^t & y_{12}^t \end{pmatrix} = \begin{pmatrix} -y_{21} & -y_{22} \\ y_{11} & y_{12} \end{pmatrix} \Leftrightarrow y_{21} = y_{21}^t \Rightarrow y_{21}$ symmetric.

$y_{22} = -y_{11}^t$, $y_{12} = y_{12}^t$

$\Rightarrow y_{12}$ symmetric.

$\Rightarrow Y = \left(\begin{array}{c|c} y_{11} & y_{12} \\ \hline y_{21} & -y_{11}^t \end{array}\right)$ has dimension $m^2 + 2\left(\frac{m^2 - m}{2} + m\right)$

$= m^2 + \cancel{8}\, m^2 - m + 2m = 2m^2 + m$

$= m(2m+1)$

So near E we have co-ords Y instead of X. For any other $A \in Sp(m,\mathbb{R})$, consider the map $X \to A^{-1}X$ taking X near A to $A^{-1}X$ near E. Thus everything we showed above is valid near all points of $Sp(m,\mathbb{R})$, thus $Sp(m,\mathbb{R})$ is a Lie group of dimension $2m^2 + m$.

orthonormal basis of V. Then $\exists F \in U(n)$ s.t. $F(e_i) = f_i \ \forall i$. Thus $U(n)$ acts transitively on $Fl(n; m_1, m_2) \Rightarrow Fl(n; m_1, m_2)$ is compact, since $U(n)$ is compact.

c) {line ⊂ plane ⊂ 3-dim affine space over $\mathbb{C}$}.

$\dim = \dim Fl(4; 2, 3)$ over $\mathbb{C}$

$= 2(4-2) + (3-2)(4-3) = 4 + 1 = 5$ over $\mathbb{C}$,

i.e. $\dim = 10$ over $\mathbb{R}$.

$SO(n) = \det^{-1}(+1)$, $O(n)_- = \det^{-1}(-1)$ open subsets in $O(n)$ since $1, -1$ are open in $\mathbb{R}^*$

3 i) a) $O(n) = \{X \in GL(n, \mathbb{R}) \mid X^t X = E\}$

$SO(n) = \{X \in GL(n, \mathbb{R}) \mid X^t X = E \text{ and } \det X = 1\}$

Have homomorphism $\det : O(n) \longrightarrow \{\pm 1\}$, since $\det(X^t X) = \det X^t \det X = (\det X)^2 = 1 \Rightarrow \det X = \pm 1$.

Clearly, $\ker \det = SO(n)$, thus we have $O(n)/SO(n) \cong \{\pm 1\}$. $\{\pm 1\}$ is a disconnected, 0-dimensional Lie group with 2 connected components. Fibres ~~of det~~ will be $A\,SO(n)$, where $\det A = -1$. Let $O(n)_- = A\,SO(n)$, where $\det A = -1$. If we can show $SO(n)$ is connected, then so shall $O(n)_-$ be and we will have found $O(n)$'s connected components.

~~Consider prove~~

We shall ~~use~~ proof by induction on $\dim SO(n) = n$. If $n = 1$, $SO(1) = 1$ is connected. Assume connectedness for $SO(2), \dots, SO(n-1)$.

~~Consider~~ Introduce Euclidean inner product on $\mathbb{R}^n$, it is given by in orthon. basis $e_1, \dots, e_n$ $(v, w) = \lambda_1 \mu_1 + \dots + \lambda_n \mu_n$ for $v = \lambda_1 e_1 + \dots + \lambda_n e_n$, $w = \mu_1 e_1 + \dots + \mu_n e_n$. Then vectors of length 1 have ~~are defined~~ by $(v, v) = \lambda_1^2 + \dots + \lambda_n^2 = 1 \cong S^{n-1}$. We consider the action of $SO(n)$ on S^{n-1} and show it is transitive.

Take e_1 of (e_1, e_1), add $e_2, \dots, e_n$ for orthonormal basis of S^{n-1}. Take another ~~$(f_1, f$~~ f of (f_1, f_1), add $f_2, \dots, f_n$ for o.n. basis of S^{n-1}. Then Set for the orthogonal transformation $\exists F \in O(n)$ s.t. $F(e_i) = f_i$. If $\det F = -1$, then change $F(e_n) = -f_n \Rightarrow \det F = 1 \Rightarrow F \in SO(n)$. Thus $SO(n)$ acts transitively on $S^{n-1} \Rightarrow S^{n-1} \cong SO(n) / SO(n)_m$ for fixed m.

Consider $SO(n)_{e_1}$. It consists of maps with matrix of the form:

$$A = \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & A_{n-1} \\ 0 & \end{pmatrix}$$

where $F(e_2), \dots, F(e_n)$ will be orthogonal to $e_1 \Rightarrow 0$'s in first row. and should be LC's of $e_2, \dots, e_n$

Clearly $\det A = \det A_{n-1} \overset{=1}{\cancel{\neq 0}}$, ~~hence~~ and ~~$X^t X = E$~~ $A^t A = E$ gives $A^t_{n-1} A_{n-1} = E_{n-1} \Rightarrow A_{n-1} \in SO(n-1)$.

Thus, by inductive hypothesis, A_{n-1} is connected to the identity $\Rightarrow A \in SO(n)_{e_1}$ is connected to the identity $\Rightarrow SO(n)_{e_1}$ is connected.

Recall we had $S^{n-1} = SO(n) / SO(n)_{e_1}$. Clearly S^{n-1} is connected since it is the $n-1$ sphere, and we just showed that $SO(n)_{e_1}$ is connected. Thus we conclude that $SO(n)$ is connected, $\Rightarrow O(n)_-$ is connected. So $O(n)$ has 2 connected components; $SO(n)$ and $O(n)_-$.

b) $U(n) = \{ X \in \overset{GL}{\cancel{Mat}}_n(\mathbb{C}) \mid \overline{X^t} X = E \}$

$SU(n) = \{ X \in GL_n(\mathbb{C}) \mid \overline{X^t} X = E \text{ and } \det X = 1 \}$.

Have $\overline{X^t} X = E \Rightarrow \det \overline{X^t} \det X = 1 \Rightarrow \overline{(\det X)^t} \det X = 1$

$\Rightarrow |\det X| = 1$. Thus $\det : U(n) \longrightarrow U(1)$, with $\ker \det = SU(n) \Rightarrow U(1) = U(n) / SU(n)$. $U(1) \cong S^1$ the circle is connected, so if we can show $SU(n)$ is connected then $U(n)$ is also connected. Fibres of ~~homom.~~ are

$A \in U(n)$ $A\,SU(n)$ where $|\det A| = 1$, $\det A \neq 1$. $(v,w) = \lambda_1 \overline{\mu_1} + \dots + \lambda_n \overline{\mu_n}$

Introduce Hermitian inner product over $\mathbb{R}^{n}$. Vectors of length 1 are given by ~~$|z|$~~ $(v,v) = |\lambda_1|^2 + \dots + |\lambda_n|^2 \cong S^{2n-1}$. Thus we consider the action of $SO(n)$ on S^{2n-1} and show it is transitive.

Take e_1 of $(e_1, e_1) = 1$, add $e_2, \dots, e_n$ $\overset{\in}{}$ $\mathbb{C}$ — $\mathbb{C}$ or $\mathbb{C}^n$ for basis of S^{2n-1}. Similarly take $f_1 \in \mathbb{C}$ of $(f_1, f_1) = 1$, add $f_2, \dots, f_n$ for basis of S^{2n-1}. Then $\exists F \in U(n)$ s.t. $F(e_i) = f_i \ \forall i$.

If $\det F = \lambda$ where $|\lambda| = 1$ but $\lambda \neq 1$, then change $F(e_n) = \frac{1}{\lambda} f_n \Rightarrow \det F = 1 \Rightarrow F \in SU(n)$. Thus $SU(n)$ acts transitively on S^{2n-1}! $\Rightarrow S^{2n-1} = SU(n) / SU(n)_m$, where m is fixed.

Now we use induction on $\dim SU(n) = n$. $SU(1) = 1 \Rightarrow SU(1)$ is connected, assume $SU(2), \dots, SU(n-1)$ are connected and prove for $SU(n)$.

Consider $SU(n)_{e_1} = \{ F \in SU(n) \mid f(e_1) = e_1 \}$. It consists of maps with matrix:

$$A = \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & A_{n-1} \\ 0 & \end{pmatrix}.$$

Clearly, $\det A = 1 = \det A_{n-1}$
$\Rightarrow \det A_{n-1} = 1$.

Also, $\overline{A^t} A = E_n \Rightarrow \overline{A^t_{n-1}} A_{n-1} = E \Rightarrow A_{n-1} \in SU(n-1)$, which by the inductive hypothesis is connected. Thus, $SU(n)_{e_1}$ is connected. ✓

Recall we had $S^{2n-1} = SU(n) / SU(n)_{e_1}$. Since S^{2n-1} is clearly connected, and we showed $SU(n)_{e_1}$ is connected, it follows that $SU(n)$ is connected.

Furthermore, since $U(1) = U(n) / SU(n)$, and $U(1) \cong S^1$ is connected, and we showed $SU(n)$ is connected, we conclude that $U(n)$ is connected. ∎

ii) $Fl(n; m_1; m_2) = U(n) / U(n)_{W_1 \subset W_2}$, $U(n)_{W_1 \subset W_2 \subset V} = \{ F \in U(n) \mid F(W_1) = W_1,\ F(W_2) = W_2 \}$.

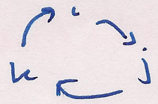
Has matrix:

$$X = \begin{pmatrix} A_{m_1, m_1} & 0 & 0 \\ 0 & B_{m_2 - m_1} & * \\ 0 & 0 & C_{n - m_2} \end{pmatrix}$$

Have $|\det X| = |\det A_{m_1}| |\det B_{m_2 - m_1}| |\det C_{n - m_2}| = 1$ ($\Rightarrow A_{m_1}, B_{m_2 - m_1}, C_{n - m_2}$ are open subsets in $U(n)$ of dim. $m_1, m_2 - m_1, n - m_2$ resp.)

Also, $\overline{X^t} X$ gives "E_n" $\overline{A^t_{m_1}} A_{m_1} = E_{m_1}$, $\overline{B^t_{m_2 - m_1}} B_{m_2 - m_1} = E_{m_2 - m_1}$, $\overline{C^t_{n - m_2}} C_{n - m_2} = E_{n - m_2} \Rightarrow$. Hence they are all connected.

It just remains to connect $*$ to the $(n - m_2)(m_2 - m_1)$ dim



O matrix, then we can connect $Fl(n;m_1,m_2)$ to E
$\Rightarrow$ Fl is connected. $\blacksquare$ check online.

4) a) Let $q_1 = a_1 + b_1 i + c_1 j + d_1 k$, $q_2 = a_2 + b_2 i + c_2 j + d_2 k$.
Then $q_1 + q_2 = (a_1+a_2) + (b_1+b_2)i + (c_1+c_2)j + (d_1+d_2)k$.
Also, $q_1 q_2 = a_1a_2 + a_1b_2 i + a_1c_2 j + a_1d_2 k + a_2b_1 i - b_1b_2$
$+ b_1c_2 k - b_1d_2 j + a_2c_1 j - c_1b_2 k - c_1c_2 + c_1d_2 i$
$+ a_2d_1 k + d_1b_2 j - d_1c_2 i - d_1d_2$
$= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + i(a_1b_2 + a_2b_1 + c_1d_2 - d_1c_2)$
$+ j(a_1c_2 - b_1d_2 + a_2c_1 + d_1b_2) + k(a_1d_2 + b_1c_2 - c_1b_2 + a_2d_1)$

Let $q = a + bi + cj + dk$. Then $\bar{q} = a - bi - cj - dk$,
and $q^{-1} = \frac{1}{q} = \frac{\bar{q}}{q\bar{q}} = \frac{\bar{q}}{|q|} = \frac{a - bi - cj - dk}{a^2+b^2+c^2+d^2}$ for $q \neq 0$.
$|q| = a^2+b^2+c^2+d^2$

b) $Sp(1) = \{q \in \mathbb{H} \mid |q| = 1\}$. $|q| = 1 \Rightarrow a^2+b^2+c^2+d^2 = 1$
clearly identifyable with the unit sphere $S^3 \subset \mathbb{R}^4$.

c) $(p,q) \in Sp(1) \times Sp(1)$, $\tau(p,q) : \mathbb{H} \longrightarrow \mathbb{H}$, $\tau(p,q)(x) = pxq^{-1} = px\bar{q}$.
$|\tau(p,q)(x)| = |pxq^{-1}| = |p||x||q^{-1}| = |x|$ hence τ preserves the modulus of x $\Rightarrow$ τ defines an orthogonal transformation of $\mathbb{H}$. Thus we have a map $\tau : Sp(1) \times Sp(1) \longrightarrow O(4)$.
Let us prove that it's a homomorphism.
Take $(p_1,q_1), (p_2,q_2) \in Sp(1) \times Sp(1)$. Then
$\tau(p_1p_2, q_1q_2) = p_1p_2 x (q_1q_2)^{-1}$. We have that $q^{-1} = \frac{\bar{q}}{|q|} = \bar{q}$ for $q \in Sp(1)$, hence $\tau(p_1p_2, q_1q_2) = p_1p_2 x \overline{q_1q_2} = p_1p_2 x \bar{q}_2\bar{q}_1$
$= p_1 \tau(p_2,q_2)(x) \bar{q}_1 = \tau(p_1,q_1)(\tau(p_2,q_2)(x))$.
Also, $\tau(1,1) = x$ $\Rightarrow$ τ is a homomorphism.

d) Really, since $Sp(1)$ is connected (it is S^3) $\Rightarrow$ $Sp(1) \times Sp(1)$ is connected,
then its image under τ must be the connected components

of the identity of $O(4)$, i.e. $\tau : Sp(1)\times Sp(1) \longrightarrow SO(4)$.

Let us find the kernel of τ. We are looking for $(p,q) \in Sp(1)\times Sp(1)$ s.t. $pxq^{-1} = x \Rightarrow px = xq$. Since τ is a homomorphism and $\mathbb{R}$-linear, it's enough to check for $x = 1, i, j, k$.

Let $p = a_1 + b_1 i + c_1 j + d_1 k$, $q = a_2 + b_2 i + c_2 j + d_2 k$.

Then $p \cdot 1 = 1 \cdot q \Rightarrow p = q$. Next for $x = i$:

$(a_1 + b_1 i + c_1 j + d_1 k)\, i = i\,(a_1 + b_1 i + c_1 j + d_1 k)$

$\Longleftrightarrow a_1 i - b_1 - c_1 k + d_1 j = a_1 i - b_1 + c_1 k - d_1 j \Rightarrow c_1 = d_1 = 0.$

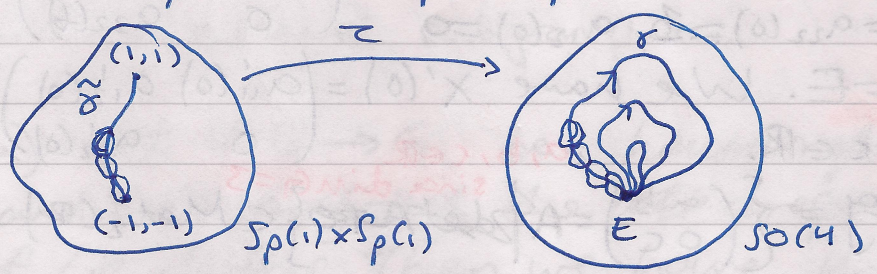
Now for $x = j$: $(a_1 + b_1 i)\, j = j\,(a_1 + b_1 i)$

$\Longleftrightarrow a_1 j + b_1 k = a_1 j - b_1 k \Longleftrightarrow b_1 = 0$. For $x = k$, $a_1 k = k a_1$ clearly holds. Now, since $p = q \in Sp(1)$, we require $p = a_1$ has $|p| = |a_1| = 1 \Rightarrow a_1 = \pm 1$.

Thus we get $\ker \tau = \{(1,1), (-1,-1)\}$, which means

$$SO(4) = (Sp(1)\times Sp(1)) / \{\pm(1,1)\}.$$

e) Consider paths in $Sp(1)\times Sp(1)$ and in $SO(4)$.



Consider closed path γ in $SO(4)$ connecting E to itself. We have that $\dim(Sp(1)\times Sp(1)) = \dim Sp(1) + \dim Sp(1)$ $= \dim S^3 + \dim S^3 = 6$

Also, $\dim SO(4) = \frac{4\cdot 3}{2} = 6$. So they both have the same dimension. Hence they are locally isomorphic, so locally we may lift path γ in $SO(4)$ to path $\tilde{\gamma}$ in $Sp(1)\times Sp(1)$.

If we assume that $SO(4)$ is simply connected, then the path γ can be deformed to a point E. Consider the lift of this continuous deformation $\tilde{\gamma}$, which is a path between $(-1,-1)$ and $(1,1)$. Since it is a continuous deformation,