Let f_n represent the nth Fibonacci number; ie: $f_n = f_{n-1} + f_{n-2}$, $f_1 = 1$, $f_2 = 1$

Let φ represent the golden ratio;
ie: $\varphi = \frac{1+\sqrt{5}}{2}$

Lemma 1: $\varphi^2 = \varphi + 1$

◁ Brute-force proof: $\varphi^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2$

$$= \frac{(1+\sqrt{5})^2}{2^2}$$

$$= \frac{(1+\sqrt{5})(1+\sqrt{5})}{4}$$

$$= \frac{1 + 2\sqrt{5} + 5}{4}$$

$$= \frac{6 + 2\sqrt{5}}{4}$$

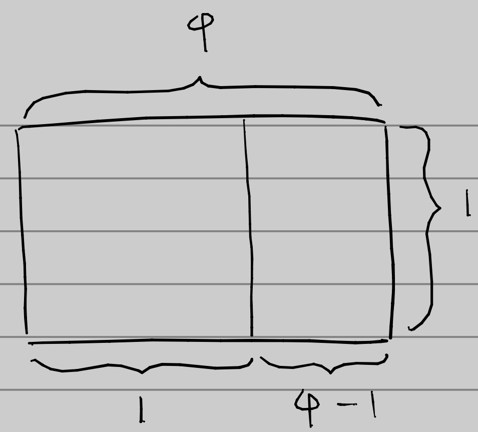
$$= \frac{3+\sqrt{5}}{2}$$

$$= \frac{1 + \sqrt{5} + 2}{2}$$

$$= \frac{1+\sqrt{5}}{2} + \frac{2}{2}$$

$$= \varphi + 1$$ ▷

◁ Geometric proof:



By def. of golden ratio, both rectangles should have the same ratio of side lengths.

ie: $\frac{\varphi}{1} = \frac{1}{\varphi - 1}$ Multiply both sides by $\varphi - 1$.

$\varphi(\varphi - 1) = 1$ Distribute φ.

$\varphi^2 - \varphi = 1$ Add φ to both sides.

$\varphi^2 = \varphi + 1$ ▷

<u>Lemma 2</u>: $\varphi^n = F_n \varphi + F_{n-1} \quad \forall n \in \{x \in \mathbb{N} \mid x \geq 2\}$

◁ Proof by induction

Base Case: $\varphi^2 = \varphi + 1$ By Lemma 1

$= F_2 \varphi + F_1$ ✓

Since $F_2 = F_1 = 1$

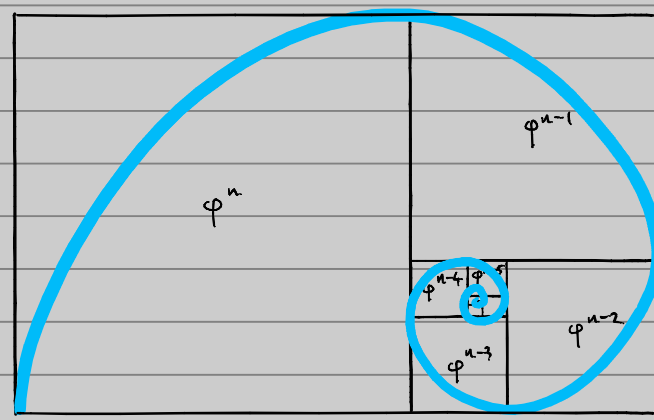
Inductive step: Suppose $\varphi^k = f_k \varphi + f_{k-1}$ for some $k \in \{x \in \mathbb{N} \mid x \geq 2\}$

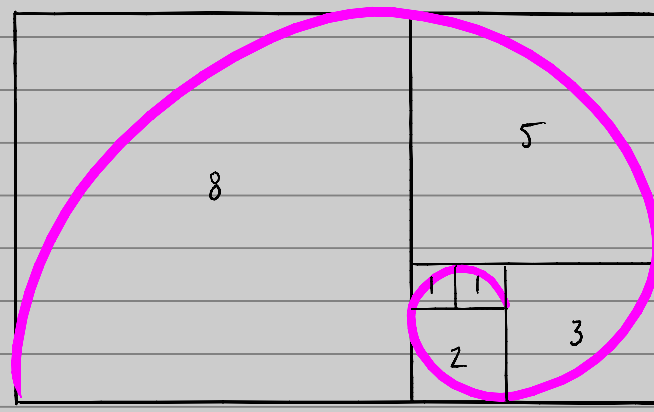
Then
$$\begin{aligned}
\varphi^{k+1} &= \varphi(\varphi^k) \\
&= \varphi(f_k \varphi + f_{k-1}) \\
&= f_k \varphi^2 + f_{k-1} \varphi \\
&= f_k(\varphi + 1) + f_{k-1} \varphi \\
&= f_k \varphi + f_k + f_{k-1} \varphi \\
&= (f_k + f_{k-1})\varphi + f_k \\
&= (f_{k+1})\varphi + f_k
\end{aligned}$$

By def. of Fibonacci #'s

$\therefore \varphi^n = f_n \varphi + f_{n-1} \quad \forall n \in \{x \in \mathbb{N} \mid x \geq 2\}$ $\triangleright$

The behaviour of a 'Fibonacci Spiral' (created by tiling $f_n \times f_n$ squares - pictured below) will approximate that of a 'Golden spiral' (created by recursively subdividing a golden rectangle into a square and a smaller golden rectangle - pictured above) as the sequence goes on.





ie: As n increases, the ratio $\frac{f_n}{f_{n-1}}$ will approach

$$\frac{\varphi^n}{\varphi^{n-1}} = \frac{\varphi(\varphi^{n-1})}{\varphi^{n-1}} = \varphi$$

◁ By Lemma 2, $\varphi^n = f_n\varphi + f_{n-1}$

$$\therefore \; f_n\varphi = \varphi^n - f_{n-1}$$

$$f_n = \frac{\varphi^n - f_{n-1}}{\varphi}$$

$$= \frac{\varphi^n}{\varphi} - \frac{f_{n-1}}{\varphi}$$

$$= \varphi^{n-1} - \frac{f_{n-1}}{\varphi}$$

Thus, $f_n = \varphi^{n-1} - \frac{f_{n-1}}{\varphi} \quad \forall n \in \{x \in \mathbb{N} \mid x \geq 2\}$

So, $f_{n-1} = \varphi^{(n-1)-1} - \frac{f_{(n-1)-1}}{\varphi}$

$$= \varphi^{n-2} - \frac{f_{n-2}}{\varphi}$$

Similarly, $f_{n-2} = \varphi^{n-3} - \frac{f_{n-3}}{\varphi}$

$$f_{n-3} = \varphi^{n-4} - \frac{f_{n-4}}{\varphi}$$

$$f_{n-4} = \varphi^{n-5} - \frac{f_{n-5}}{\varphi}$$

$$\vdots$$

$\therefore$ By substitution,

$$f_n = \varphi^{n-1} - \frac{1}{\varphi}\left(\varphi^{n-2} - \frac{f_{n-2}}{\varphi}\right)$$

$$= \varphi^{n-1} - \frac{\varphi^{n-2}}{\varphi} + \frac{f_{n-2}}{\varphi^2}$$

$$= \varphi^{n-1} - \varphi^{n-3} + \frac{f_{n-2}}{\varphi^2}$$

$$= \varphi^{n-1} - \varphi^{n-3} + \frac{1}{\varphi^2}\left(\varphi^{n-3} - \frac{f_{n-3}}{\varphi}\right)$$

$$= \varphi^{n-1} - \varphi^{n-3} + \frac{\varphi^{n-3}}{\varphi^2} - \frac{f_{n-3}}{\varphi^3}$$

$$= \varphi^{n-1} - \varphi^{n-3} + \varphi^{n-5} - \frac{f_{n-3}}{\varphi^3}$$

$$= \varphi^{n-1} - \varphi^{n-3} + \varphi^{n-5} - \frac{1}{\varphi^3}\left(\varphi^{n-4} - \frac{f_{n-4}}{\varphi}\right)$$

$$= \varphi^{n-1} - \varphi^{n-3} + \varphi^{n-5} - \frac{\varphi^{n-4}}{\varphi^3} + \frac{f_{n-4}}{\varphi^4}$$

$$= \varphi^{n-1} - \varphi^{n-3} + \varphi^{n-5} - \varphi^{n-7} + \frac{f_{n-4}}{\varphi^4}$$

$$\vdots$$

$$= \varphi^{n-1} - \varphi^{n-3} + \ldots + (-1)^{k+1}\varphi^{n-(2k-1)} + (-1)^k \frac{f_{n-k}}{\varphi^k}$$

This pattern will continue until the value of k reaches $n-1$, since f_1 is the minimum term in the Fibonacci sequence.

$$\therefore f_n = \varphi^{n-1} - \varphi^{n-3} + \ldots + (-1)^{(n-1)+1} \varphi^{n-(2(n-1)-1)} + (-1)^{n-1} \frac{f_{n-(n-1)}}{\varphi^{n-1}}$$

$$= \varphi^{n-1} - \varphi^{n-3} + \ldots + (-1)^{n} \varphi^{n-(2n-3)} + (-1)^{n-1} \frac{f_{n-n+1}}{\varphi^{n-1}}$$

$$= \varphi^{n-1} - \varphi^{n-3} + \ldots + (-1)^{n} \varphi^{-n+3} + (-1)^{n-1} \frac{f_1}{\varphi^{n-1}}$$

$$= \varphi^{n-1} - \varphi^{n-3} + \ldots + (-1)^{n} \varphi^{-(n-3)} + (-1)^{n-1} \frac{1}{\varphi^{n-1}}$$

$$= \varphi^{n-1} - \varphi^{n-3} + \ldots + \frac{(-1)^{n}}{\varphi^{n-3}} + \frac{(-1)^{n-1}}{\varphi^{n-1}}$$

ie: $f_2 = \varphi - \frac{1}{\varphi} = 1$

$f_3 = \varphi^2 - 1 + \frac{1}{\varphi^2} = 2$

$f_4 = \varphi^3 - \varphi + \frac{1}{\varphi} - \frac{1}{\varphi^3} = 3$

$f_5 = \varphi^4 - \varphi^2 + 1 - \frac{1}{\varphi^2} + \frac{1}{\varphi^4} = 5$

$f_6 = \varphi^5 - \varphi^3 + \varphi - \frac{1}{\varphi} + \frac{1}{\varphi^3} - \frac{1}{\varphi^5} = 8$

$\vdots$

Or, $f_n = \sum_{i=1}^{n} (-1)^{i+1} \varphi^{n-(2i-1)}$

$$= \sum_{i=1}^{n} (-1)^{i+1} \varphi^{n-2i+1} \quad \forall n \in \{x \in \mathbb{N} \mid x \geq 2\}$$

$$\therefore f_{n-1} = \sum_{i=1}^{n-1} (-1)^{i+1} \varphi^{(n-1)-2i+1}$$
$$= \sum_{i=1}^{n-1} (-1)^{i+1} \varphi^{n-2i}$$

Finally,

$$\lim_{n\to\infty} \frac{f_n}{f_{n-1}} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} (-1)^{i+1} \varphi^{n-2i+1}}{\sum_{i=1}^{n-1} (-1)^{i+1} \varphi^{n-2i}}$$

$$= \lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} (-1)^{i+1} \varphi^{n-2i+1} + (-1)^{n+1} \varphi^{n-2n+1}}{\sum_{i=1}^{n-1} (-1)^{i+1} \varphi^{n-2i}}$$

$$= \lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} (-1)^{i+1} \varphi(\varphi^{n-2i}) + (-1)^{n+1} \varphi^{-n+1}}{\sum_{i=1}^{n-1} (-1)^{i+1} \varphi^{n-2i}}$$

$$= \lim_{n\to\infty} \frac{\varphi \sum_{i=1}^{n-1} (-1)^{i+1} (\varphi^{n-2i}) + \frac{(-1)^{n+1}}{\varphi^{n-1}}}{\sum_{i=1}^{n-1} (-1)^{i+1} \varphi^{n-2i}}$$

$$= \frac{\lim_{n\to\infty} \varphi \sum_{i=1}^{n-1} (-1)^{i+1} (\varphi^{n-2i}) + \lim_{n\to\infty} \frac{(-1)^{n+1}}{\varphi^{n-1}}}{\lim_{n\to\infty} \sum_{i=1}^{n-1} (-1)^{i+1} (\varphi^{n-2i})}$$

$$= \frac{\varphi \lim_{n\to\infty} \sum_{i=1}^{n-1} (-1)^{i+1} (\varphi^{n-2i}) + 0}{\lim_{n\to\infty} \sum_{i=1}^{n-1} (-1)^{i+1} (\varphi^{n-2i})}$$

$$= \varphi \lim_{n\to\infty} \left(\frac{\sum_{i=1}^{n-1} (-1)^{i+1} (\varphi^{n-2i})}{\sum_{i=1}^{n-1} (-1)^{i+1} (\varphi^{n-2i})} \right)$$

$$= \varphi (1)$$

$$\therefore \lim_{n \to \infty} \frac{f_n}{f_{n-1}} = \varphi$$

ie: The ratio $\frac{f_n}{f_{n-1}}$ approaches the golden ratio as n increases. $\triangleright$